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An uniform approach to give high-precision eigenvalue estimation for self-adjoint eigenvalue problems

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Various eigenvalue problems

- Laplace operator:

$$-\Delta u = \lambda u \text{ in } \Omega \quad \oplus \quad \text{boundary condition}$$

- Bi-harmonic operator:

$$\Delta^2 u = \lambda u \text{ or } \Delta^2 u = -\lambda \Delta u \text{ in } \Omega \quad \oplus \quad \text{boundary condition}$$

- Eigenvalue problems for Stokes equations:

$$\begin{cases} -\Delta u + \nabla p = \lambda u \\ \text{div } u = 0 \end{cases} \text{in } \Omega \quad \oplus \quad \text{boundary condition}$$

- Eigenvalue problem for Maxwell's equation:

  Find $E \in H_0(\textbf{rot};\Omega)$ and $\lambda \in R$, s.t.,

$$(\textbf{rot}E, \textbf{rot}F) = \lambda(E, F) \quad \forall F \in H_0(\textbf{rot};\Omega)\ .$$

Objective

Explicit eigenvalue bounds for eigenvalue problems

Eigenvalue problem defined by abstract variational form:

$$\text{Find } u \in V \text{ and } \lambda > 0 \text{ s.t.} \quad M(u,v) = \lambda\, N(u,v), \quad \forall v \in V$$

where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

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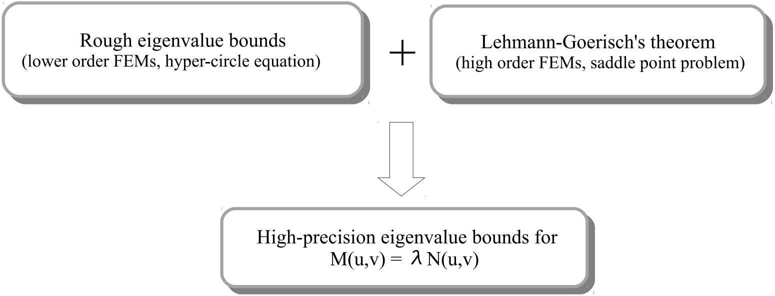
where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms to be defined.

- Such an abstract problem will include eigenvalue problems of the Laplace operator, the Bi-harmonic operator, the Stokes equation and the Maxwell equation.
- This day, we consider operators, Δ, Δ^2, in 1D, 2D and 3D space.

Outline

1. Framework for high-precision eigenvalue bounds.
2. The eigenvalue problem of Laplace operators.
   - Crouzeix-Raviart FEM space
   - Explicit error estimation for Crouzeix-Raviart interpolation
3. The eigenvalue problem of Bi-harmonic operators.
   - Fujino-Morley FEM space
   - Explicit error estimation for Fujino-Morley interpolation
4. Sharpen the bounds by applying Lehmann-Goerisch's theorem
5. Computation results

The framework for high-precision eigenvalue bounds



Rough lower eigenvalue bounds by using lower order FEMs

Lower and upper eigenvalue bounds

Preparation

- V: Hilbert space of real functions defined over domain Ω.
- V^h: FEM space over the triangulation $\mathcal{T}^h$ for Ω; V^h may not be a subspace of V.

Assumption

A1 $M(u,v)$, $N(u,v)$ are symmetric bilinear forms over V and V^h; $M(u,u) \geq 0$, $N(u,u) \geq 0$; $N(u,u) = 0$ implies $u = 0$.

- Define $|\cdot|_M := \sqrt{M(\cdot,\cdot)}$, $|\cdot|_N := \sqrt{N(\cdot,\cdot)}$.

A2 There exist sequences $\{\phi_i\}_{i\in\mathbf{N}}$ and non-decreasing $\{\lambda_i\}_{i\in\mathbf{N}}$ such that $\phi_i \in V$, $\lambda_i \in \mathbf{R}$, $N(\phi_i, \phi_k) = \delta_{ik}$ for $i, k \in \mathbf{N}$,

$$M(f, \phi_i) = \lambda_i N(f, \phi_k) \qquad \text{for all } f \in V, i \in \mathbf{N}. \tag{1}$$

$$N(f, f) = \sum_{i=1}^{\infty} (N(f, \phi_i))^2 \qquad \text{for all } f \in V, i \in \mathbf{N}. \tag{2}$$

Min-max principle

Min-max principle

Define Rayleigh quotient $R(u)$ over V and V^h.

$$R(u) := M(u,u)/N(u,u)\,.$$

The min-max principle tells that

$$\lambda_k = \min_{\dim(H)=k; H\subset V} \max_{u\in H} R(u)$$

Upper eigenvalue bounds

Eigenvalue problem in V^h

Let $(\lambda_{h,k}, \phi_{h,k})_{k=1,\cdots,n}$ $(\lambda_{h,k} \le \lambda_{h,k+1})$ be the eigen-pairs such that,

$$M(v_h, \phi_{h,k}) = \lambda_{h,k} N(v_h, \phi_{h,k}) \quad \forall v_h \in V^h .$$

Theorem (Upper eigenvalue bounds)

If $V^h \subset V$, then an upper bound for λ_k is given as,

$$\lambda_k \le \lambda_{h,k} .$$

Upper eigenvalue bounds

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If $V^h \subset V$, then an upper bound for λ_k is given as,

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Proof:
Let $E_{h,k}$ be the space spanned by $\{\phi_{h,1}, \cdots, \phi_{h,k}\}$. Then,

$$\lambda_k \le \max_{u \in E_{h,k}} R(u) = \lambda_{h,k} .$$

Upper eigenvalue bounds

Eigenvalue problem in V^h

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- In the field of finite element method (FEM), a V^h satsifying $V^h \subset V$ is called *conforming FEM space (適合有限要素空間)*.
- For eigenvalue problem associated with high-order derivatives, $V^h (\subset V)$ is not easy to construct.

Theorem for lower eigenvalue bounds

Theorem 1

Let $P_h : V \to V^h$ be a projection satisfying

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h$$

Moreover, suppose that an error estimation for P_h is given as,

$$|u - P_h u|_N \le C_h |u - P_h u|_M \, .$$

Assertion: The lower bounds for eigenvalues are given as,

$$\lambda_{h,k}/(1 + \lambda_{h,k} C_h^2) \le \lambda_k \quad (k = 1, 2, \cdots, n) \, .$$

Main proof for Theorem 1

With current assumption, the min-max principle holds for eigenvalues:

$$\lambda_{h,k} = \min_{\dim(H)=k; H\subset V^h} \max_{u\in H} R(u)$$

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Let E_k be the space spanned by $\{\phi_1, \cdots, \phi_k\}$. By showing that dim($P_h E_k$)=k, the following inequality holds,

$$\lambda_{h,k} \le \max_{v_h\in P_h E_k} R(v_h) = \max_{v\in E_k} R(P_h v) \qquad .$$

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$$R(P_h v) = \frac{|P_h v|_M^2}{|P_h v|_N^2}$$

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With the error estimation for P_h, we have,

$$R(P_h v) \le \frac{\lambda_k}{1-C_h^2\lambda_k} \qquad .$$

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Inspired by the result in

- *X. Liu and S. Oishi. Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal., 51(3):1634–1654, 2013.*

(Only conforming FEM space $V^h(\subset V)$ is considered.)

Two tasks in the application of Theorem 1

1) Selection of proper FEM space V^h and the projection P_h:

$$M(u - P_h u, v_h) = 0, \quad \text{for all } v_h \in V^h .$$

2) Explicit error estimation for P_h:

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- A locally defined interpolation operator Π_h that is also a projection operator will be a good candidate for P_h.
- That is, on each element K of triangulation $\mathcal{T}^h$,

$$(P_h u)|_K = \Pi_h (u|_K) .$$

2. Eigenvalue problem of Laplace operator

2. Eigenvalue problem of Laplace operator

- Two kind of projection P_h's will be considered.

Eigenvalue problem of Laplace operators

Eigenvalue problem for 2rd order differential operator

Assumption: Ω is a simply connected bounded domain.

$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial\Omega,$$

Definition of operators: (2D)

- $\nabla u = (u_x, u_y)$ for u being a scalar function.
- $\text{div}\, p = p_{1,x} + p_{2,y}$ for $p = (p_1, p_2)$ being a vector function.
- $\Delta u = \text{div} \cdot \nabla u = u_{xx} + u_{yy}$.

Eigenvalue problem of Laplace operators

Eigenvalue problem for 2rd order differential operator

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$$-\Delta u = \lambda u, \quad u = 0 \text{ on } \partial\Omega,$$

Variational formulation:

Let $V := \{v \in H^1(\Omega) |\ u = 0 \text{ on } \partial\Omega\}$.

$$\text{Find } u \in V \text{ and } \lambda \geq 0 \text{ s.t.} \quad \int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega uv dx \quad \forall v \in V\ .$$

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where $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are bilinear forms over V:

$$M(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad N(u,v) = \int_\Omega uv dx\,.$$

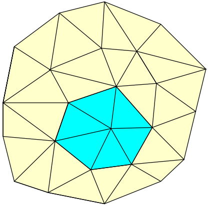
Conforming FEM space

Eigenvalue problem in FEM spaces V^h

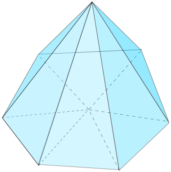
Lagrange FEM space: $V^h(\subset V)$

Let $\mathcal{T}^h$ be a triangulation of domain Ω. The function space V^h over $\mathcal{T}^h$ is consisted of function v_h such that,

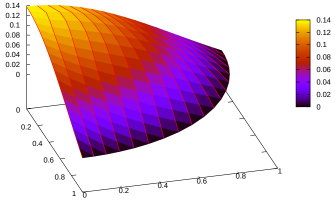
1) $v_h|_K$ is linear function on each element $K \in \mathcal{T}^h$;
2) v_h is a continuous function over Ω.



Triangulation $\mathcal{T}^h$ of domain



Base function



Sample function $u_h \in V^h$

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1) $v_h|_K$ is linear function on each element $K \in \mathcal{T}^h$;
2) v_h is a continuous function over Ω.

The bilinear forms $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ over V^h:

$$M(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx \,.$$

Eigenvalue problem in V^h: Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$

Error estimation for $P_h : V \to V^h$

A priori error estimation (事前誤差評価)

Given $f \in L_2(\Omega)$, let $u \in H_0^1(\Omega)$ and $P_h u \in V^h$ be the solutions of variational problems below, respectively,

$$M(u, v) = N(f, v) \quad \text{for } v \in H_0^1(\Omega)\,, \quad M(P_h u, v_h) = N(f, v_h) \quad \text{for } v_h \in V^h\,.$$

We have an error estimate as below,

$$\|u - P_h u\|_{L_2} \le C_h \|\nabla(u - P_h u)\|_{L_2} \le C_h^2 \|f\|_{L_2}\,. \tag{3}$$

where C_h is a quantity only depending on the mesh triangulation and domain shape.

- *X. Liu and S. Oishi, Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal., 51(3):1634–1654, 2013,*

Remarks on evaluation of C_h

- For convex domain, the solution u corresponding for given f is regular, that is, $u \in H^2(\Omega)$. The constant C_h is easy to obtain.
- For non-convex domain, u may have a singularity, that is, $u \notin H^2(\Omega)$. A new "Hypercircle equation method" is developed to give an estimation of C_h.

Lower eigenvalue bounds based on Theorem 1

Setting for application of Theorem 1

- $V = H_0^1(\Omega)$;
- V^h: Lagrange conforming FEM space ($V^h \subset V$);
- $M(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$;
- $N(u,v) := \int_\Omega uvdx$;
- Projection P_h:

$$M(u - P_h u, v_h) = 0 \text{ for } v_h \in V^h .$$

- Error estimation for P_h:

$$|u - P_h u|_N \leq C_h |u - P_h u|_M$$

Computation results based on conforming FEMs

Example I: unit isosceles right triangle domain

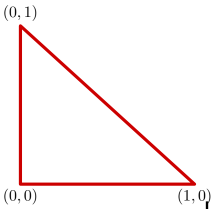
Function space: $V = H_0^1(\Omega), \quad \|u\|_V^2 = (\nabla u, \nabla u)_{L_2(\Omega)}$.
Problem: Find $u \in H_0^1(\Omega)$ and $\lambda > 0$ such that,

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma .$$

Constant values: $h = 1/32$, $C_1 \le 0.493$, Thus $M := 0.0155$

Exact values:

$$\lambda_1 = 5\pi^2 \approx 49.348, \quad \lambda_2 = 10\pi^2 \approx 98.696, \quad \lambda_3 = 13\pi^2 \approx 128.30$$

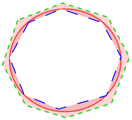


λ_i	lower	upper	rel. err
1	48.9488	49.5525	0.01225
2	96.4497	99.6328	0.03238
3	121.8806	129.7290	0.06239
4	153.8131	170.3116	0.1018
5	171.9199	201.5760	0.15888

Lower and upper bounds of eigenvalues

Example II: Square with hole

Ω: Square $(0,\pi)\times(0,\pi)$ excluding hole surrounding by $sin^2(x)+sin^2(y)=3/2$.
Exact value: $\lambda_1=10$. M := 0.0618 (outter polygon), 0.0634 (inner polygon).



λ_i	lower	upper	rel. err.
1	9.611519	10.088400	0.048414
2	9.973375	10.486400	0.050150
3	9.974390	10.487100	0.050115
4	10.449044	11.010400	0.052318
5	15.081232	16.405400	0.084110

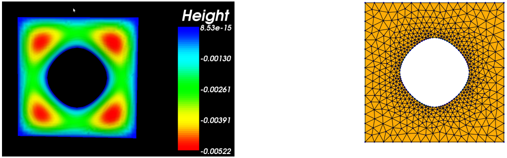


Figure : First eigenfunction and mesh

Example III: Mixed boundary on square with crack

Let Ω be unit square with crack $\{(x, 0.5) | 0 < x < 0.5\}$.

$$\partial_D \Omega = \partial\Omega \cap \{y = 1 \text{ or } y = 0, \text{ or } x = 1\}, \ \partial_N \Omega = \partial\Omega / \partial_D \Omega$$

Problem:

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial_D \Omega, \quad \partial u / \partial n = 0 \text{ on } \partial_N \Omega$$

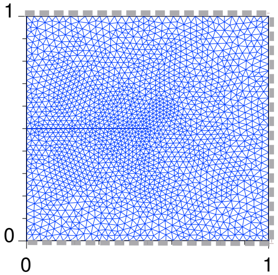
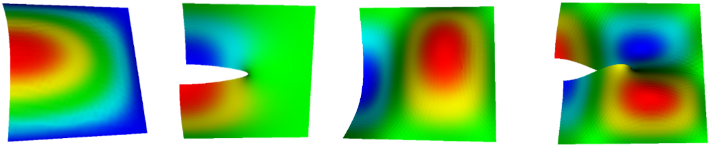


Table : Eigenvalue estimation (non-uniform mesh)

Example III: Mixed boundary on square with crack

- Eigenvalue estimation: (non-uniform mesh) $M = 0.027$

λ_i	lower	upper	relative.
1	12.233	12.343	0.009
2	16.087	16.276	0.012
3	31.392	32.119	0.022
4	51.049	52.998	0.037
5	68.241	71.768	0.050



Eigenfunctions for leading 4 eigenvalues

Non-conforming FEM

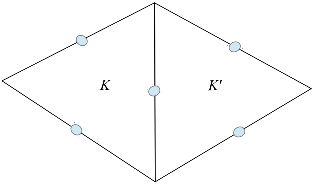
Eigenvalue problem in FEM spaces V^h

Crouzeix-Raviart FEM space: $V^h (\not\subset V)$

The function v_h of V^h satisfies,

1) v_h is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Function v_h is only continuous on the mid-points of interior edges.



Eigenvalue problem in FEM spaces V^h

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1) v_h is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Extend the bilinear forms M and N from V to V^h:

$$M(u_h, v_h) = \sum_{K \in \mathcal{T}^h} \int_K \nabla u_h \cdot \nabla v_h dx, \quad N(u_h, v_h) = \int_\Omega u_h v_h dx \,.$$

Eigenvalue problem in FEM spaces V^h

Crouzeix-Raviart FEM space: $V^h (\not\subset V)$

The function v_h of V^h satisfies,

1) v_h is linear on each element $K \in \mathcal{T}^h$;
2) $\int_e v_h ds$ is continuous on interior edges; $\int_e v_h ds = 0$ on boundary edges;

Eigenvalue problem in V^h

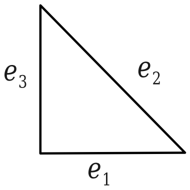
Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$

The Crouzeix-Raviart interpolation Π_h

On triangle element K, $(\Pi_h u)|_K$ is a linear function such that

$$\int_{e_i} u - \Pi_h u \ ds = 0 \quad (i = 1, 2, 3).$$



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Important property of Π_h

For $u \in H^1(\Omega)$,

$$\int_K \nabla(u - \Pi_h u) \cdot \nabla v_h dx = 0, \quad \forall v_h \in P^1(K) \, .$$

Hint:

$$\int_K \nabla(u - \Pi_h u) \cdot \nabla v_h dx = \int_{\partial K} (u - \Pi_h u) v_h ds - \int_K (u - \Pi_h u) \cdot \Delta v_h dx$$

The Crouzeix-Raviart interpolation Π_h

On triangle element K, $(\Pi_h u)|_K$ is a linear function such that

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Hence

$$\sum_{K \in \mathcal{T}^h} \int_K \nabla(u - \Pi_h u) \cdot \nabla v_h dx = 0, \quad \forall v_h \in V^h \qquad .$$

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Hence

$$\sum_{K \in \mathcal{T}^h} \int_K \nabla(u - \Pi_h u) \cdot \nabla v_h dx = 0, \quad \forall v_h \in V^h \qquad .$$

Thus,

$$M(u - \Pi_h u, v_h) = 0, \quad \forall v_h \in V^h \,.$$

Interpolation error estimation for Π_h

On each element K, we consider the error estimation by using $C_e(K)$:

$$\|u - \Pi_h u\|_{0,K} \le C_e(K) |u - \Pi_h u|_{1,K} \quad \text{for } u \in H^1(K)\,.$$

where $|\cdot|_{d,K}$ denotes the semi-norm of Sobolev function space $H^d(K)$.

Interpolation error estimation for Π_h

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where $|\cdot|_{d,K}$ denotes the semi-norm of Sobolev function space $H^d(K)$.

Error constant $C_e(K)$

Over K of $\mathcal{T}^h$, define $V_e(K)$ by

$$V_e(K) := \{v \in H^1(K) \mid \int_{e_i} v ds = 0,\ \ i = 1,2,3\}\,.$$

Constant $C_e(K)$ over $V_e(K)$:

$$C_e(K) := \sup_{u \in V_e(K)} \frac{\|u\|_{0,K}}{|u|_{1,K}}\,,$$

Upper bound of $C_e(K)$

The eigenvalue problem for $C_e(K)$: Over K,

$$-\Delta u = \lambda u, \quad \left.\frac{\partial u}{\partial n}\right|_{e_i} = c_i \quad (c_i : \text{to be determined})$$

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- For unit isoceles right triangle $\hat{K}$, an easy-to-obtain lower bound:

$$C_e(\hat{K}) \le \frac{1}{\pi}\,.$$

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- Approximate computation shows that,

$$C_e(\hat{K}) \approx 0.2377 < \frac{1}{\pi} \approx 0.3183\,.$$

Error estimation for projection Π_h

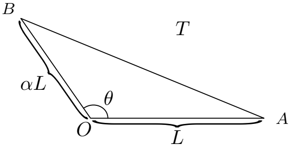
Upper bound for $C_e(K)$

For general triangle K with the maximum inner angle as θ and the second longest edge length as L,

$$C_e(K) \le \frac{L}{\pi}\sqrt{1+|\cos\theta|}$$

Hence,

$$\|u-\Pi_h u\|_{0,K} \le \frac{L}{\pi}\sqrt{1+|\cos\theta|}\,|u-\Pi_h u|_{1,K}$$



Remarks on Crouzeix-Raviart interpolation Π_h

- Even for non-convex domains, the interpolation Π_h is well-defined for $u \in H^1(\Omega)$. Thus there is no additional efforts needed for problems with singularities.
- Easy to deal with boundary conditions.

Lower eigenvalue bounds based on Theorem 1

Setting for application of Theorem 1

- $V = H_0^1(\Omega)$;
- V^h: Crouzeix-Rarviart FEM space ($V^h \not\subset V$);
- $M(u,v) := \sum_{K\in\mathcal{T}^h} \int_K \nabla u \cdot \nabla v dx$;
- $N(u,v) := \int_\Omega uv dx$;
- Projection $P_h := \Pi_h$:

$$M(u - P_h u, v_h) = 0 \text{ for } v_h \in V^h .$$

- Error estimation for P_h:

$$|u - P_h u|_N \le C_h |u - P_h u|_M \quad \left(C_h := \max_{K\in\mathcal{T}^h} C_e(K) \right)$$

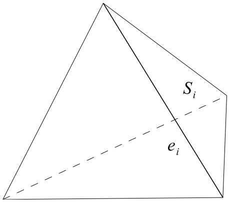
General Crouzeix-Raviart interpolation operators

Crouzeix-Raviart type interpolation in 3D

Extension of Crouzeix-Raviart interpolation in 3D
Let K be a tetrahedron with surfaces S_i, $i = 1, 2, 3, 4$.

For $u \in H^1(K)$, $\Pi_h u \in P^1(K)$ is determined by

$$\int_{S_i} (u - \Pi_h u) dS = 0, \quad i = 1, 2, 3\,.$$



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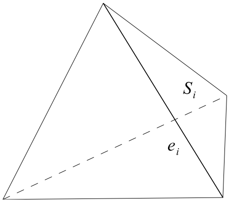

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Property

For any $v \in P^1(K)$,

$$\int_K \nabla(u - \Pi_h u) \cdot \nabla v dX = 0\,.$$

Crouzeix-Raviart type interpolation in 1D

Extension of Crouzeix-Raviart interpolation in 1D

Let I be an interval with two vertices a and b.

For $u \in H^1(I)(\subset C^0(I))$, $\Pi_h u \in P^1(I)$ is determined by

$$(\Pi_h u)(a) = u(a), \quad (\Pi_h u)(b) = u(b)\ .$$

Crouzeix-Raviart type interpolation in 1D

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$$(\Pi_h u)(a) = u(a), \quad (\Pi_h u)(b) = u(b)\ .$$

Property

For any $v \in P^1(I)$,

$$\int_I (u - \Pi_h u)^{(1)} \cdot v^{(1)} ds = 0\ .$$

3. Eigenvalue problem of Bi-harmonic operator

Buckling eigenvalue problem

Eigenvalue problem for 4th order differential operator

Assumption: Ω is a simply connected bounded domain.

$$\Delta\Delta u = -\lambda\Delta u, \quad u = \partial u/\partial n = 0 \text{ on } \partial\Omega,$$

Definition of operators: (2D)

- $\nabla u = (u_x, u_y)$ for u being a scalar function.
- $D^2 u = (u_{xx}, u_{xy}, u_{yx}, u_{yy})$ for u being a scalar function.
- $\text{div} p = p_{1,x} + p_{2,y}$ for $p = (p_1, p_2)$ being a vector function.
- $\Delta u = \text{div} \cdot \nabla u = u_{xx} + u_{yy}$
- $\text{curl}\, p = p_{2,x} - p_{1,y}$ for $p = (p_1, p_2)$.
- $\text{curl}\, u = (u_y, -u_x)$ for u being a scalar function.

Buckling eigenvalue problem

Eigenvalue problem for 4th order differential operator

Assumption: Ω is a simply connected bounded domain.

$$\Delta\Delta u = -\lambda \Delta u, \quad u = \partial u / \partial n = 0 \text{ on } \partial\Omega,$$

Variational formulation:

$V := \{v \in H^2(\Omega) |\ u = \partial u / \partial n = 0 \text{ on } \partial\Omega\}$.

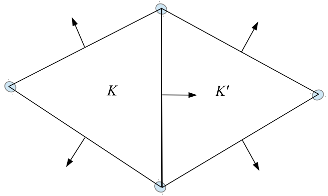
$$\text{Find } u \in V \text{ and } \lambda \geq 0 \text{ s.t.} \quad (D^2 u, D^2 v)_\Omega = \lambda (\nabla u, \nabla v)_\Omega \quad \forall v \in V$$

Eigenvalue problem in FEM spaces V^h

Fujino-Morley FEM space: $V^h (\not\subset V)$

The function v_h of V^h satisfies,

1) $v_h|_K \in P^2(K)$ on each element $K \in \mathcal{T}^h$;
2) $\int_e \frac{\partial v_h}{\partial n} ds$ is continuous on interior edges; $\int_e \frac{\partial v_h}{\partial n} ds = 0$ on boundary edges;
3) v_h is continuous on interior nodes;. $v_h = 0$ on boundary nodes.



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3) v_h is continuous on interior nodes;. $v_h = 0$ on boundary nodes.

Define bilinear forms $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ over V^h:

$$M(u_h, v_h) = \sum_{K\in\mathcal{T}^h} (D^2 u_h, D^2 v_h)_K, \quad N(u_h, v_h) = \sum_{K\in\mathcal{T}^h} (\nabla u_h, \nabla v_h)_K .$$

- For $u, v \in H^2(\Omega)$, $M(u,v) = (D^2 u, D^2 v)$, $N(u,v) = (Du, Dv)$.

Eigenvalue problem in FEM spaces V^h

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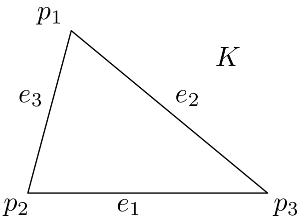
Find $u_h \in V^h$ and $\lambda_h \geq 0$ s.t.

$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$

The Fujino-Morley interpolation Π_h

On triangle element K, $(\Pi_h u)|_K$ is a quadratic function such that

$$(u - \Pi_h u)(p_i) = 0, \quad \int_{e_i} \nabla(u - \Pi_h u) \cdot n \ ds = 0 \quad (i = 1, 2, 3).$$



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Important property of Π_h

For $u \in H^2(\Omega)$,

$$(D^2(u-\Pi_h u), D^2 v_h)_K = 0, \quad \forall v_h \in V^h \ .$$

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Thus,

$$M(u - \Pi_h u, v_h) = 0, \quad \forall v_h \in V^h .$$

Interpolation error estimation for Π_h

On each element K, we consider the error estimation by using $C_1(K)$:

$$|u - \Pi_h u|_{1,K} \le C_1(K)|u - \Pi_h u|_{2,K} \quad \text{for } u \in H^2(K)\,.$$

where $|\cdot|_{d,K}$ denotes the semi-norm of Sobolev function space $H^d(K)$.

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where $|\cdot|_{d,K}$ denotes the semi-norm of Sobolev function space $H^d(K)$.

Error constant $C_1(K)$

Over K of $\mathcal{T}^h$, define $W(K)$ by

$$W(K) := \{u \in H^2(K) \mid u(p_i) = 0, \int_{e_i} \frac{\partial u}{\partial n} ds = 0, \ i = 1,2,3\} \,.$$

Constant $C_1(K)$ over $W(K)$:

$$C_1(K) := \sup_{u \in W(K)} \frac{|u|_{1,K}}{|u|_{2,K}} \,,$$

Babuska-Aziz's technique for upper bound of $C_1(K)$

Notice

$$C_1(K)^2 = \sup_{u \in W(K)} \frac{\|u_x\|_0^2 + \|u_y\|_0^2}{|u_x|_1^2 + |u_y|_1^2} \le \sup_{u \in W(K)} \max\left(\frac{\|u_x\|_0^2}{|u_x|_1^2}, \frac{\|u_y\|_0^2}{|u_y|_1^2}\right)$$

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About u_x, u_y:

Notice that $u(p_i)=0$, we have for each $i=1,2,3$,

$$\int_{e_i} (u_x, u_y)\cdot n dx = \int_{e_i} (u_x, u_y)\cdot \tau dx = 0,$$

where τ is the unit tangent vector and n the unit norm vector along e_i. Thus

$$\int_{e_i} u_x ds = \int_{e_i} u_y ds = 0 \quad (i=1,2,3)$$

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Notice

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Recall the definition of constant $C_e(K)$:

$$C_e(K) := \sup_{v \in V_e(K)} \frac{\|v\|_{0,K}}{|v|_{1,K}} ,$$

where

$$V_e(K) := \{v \in H^1(K) \mid \int_{e_i} vds = 0, i = 1,2,3\}$$

Then:

$$\|u_x\|_0 \le C_e(K)|u_x|_1, \quad \|u_y\|_0 \le C_e(K)|u_y|_1 .$$

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Notice

$$C_1(K)^2 = \sup_{u\in W(K)} \frac{\|u_x\|_0^2 + \|u_y\|_0^2}{|u_x|_1^2 + |u_y|_1^2} \le \sup_{u\in W(K)} \max\left(\frac{\|u_x\|_0^2}{|u_x|_1^2}, \frac{\|u_y\|_0^2}{|u_y|_1^2} \right)$$

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Then:

$$\|u_x\|_0 \le C_e(K)|u_x|_1, \quad \|u_y\|_0 \le C_e(K)|u_y|_1 .$$

$$C_1(K) \le C_e(K)$$

Babuska-Aziz's technique for upper bound of $C_1(K)$

- For unit isoceles right triangle $\hat{K}$, an easy-to-obtain lower bound:

$$C_e(\hat{K}) \le \frac{1}{\pi}\,.$$

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- For unit isoceles right triangle $\hat{K}$, an easy-to-obtain lower bound:

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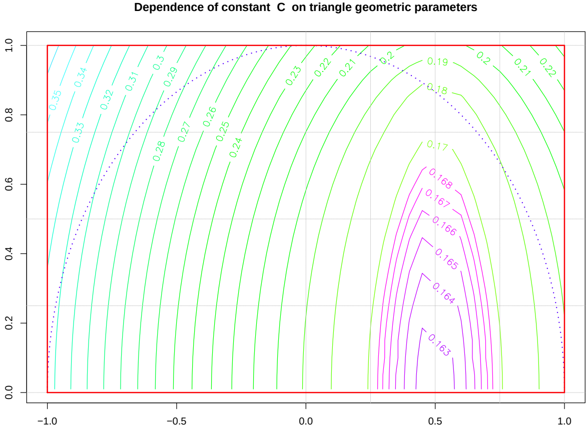
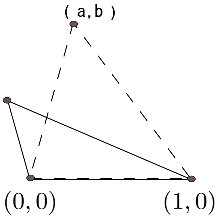
- Approximate computation shows that,

$$C_1(\hat{K}) \approx 0.2338 < C_e(\hat{K}) \approx 0.2377 < \frac{1}{\pi} \approx 0.3183\,.$$

Contour lines of the constant $C_1(K)$





Error estimation for projection Π_h

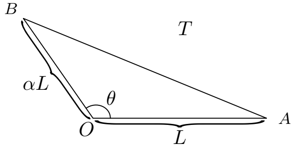
Upper bound for $C_1(K)$

For general triangle K with the maximum inner angle as θ and the second longest edge length as L,

$$C_1(K) \le \frac{L}{\pi}\sqrt{1+|\cos\theta|}$$

Hence,

$$|u-\Pi_h u|_{1,K} \le \frac{L}{\pi}\sqrt{1+|\cos\theta|}\,|u-\Pi_h u|_{2,K}$$



Lower eigenvalue bounds based on Theorem 1

Setting for application of Theorem 1

- $V = H_0^2(\Omega)$;
- V^h: Fujino-Morley FEM space ($V^h \not\subset V$);
- $M(u,v) := \sum_{K\in\mathcal{T}^h} (D^2 u, D^2 v)_K$;
- $N(u,v) := \sum_{K\in\mathcal{T}^h} (\nabla u, \nabla v)_K$;
- Projection $P_h := \Pi_h$:

$$M(u - P_h u, v_h) = 0 \text{ for } v_h \in V^h \,.$$

- Error estimation for P_h:

$$|u - P_h u|_N \le C_h |u - P_h u|_M \quad \left(C_h := \max_{K\in\mathcal{T}^h} C_1(K) \right)$$

Lower eigenvalue bounds based on Theorem 1

Lower eigenvalue bounds

From Theorem 1, we have,

$$\frac{\lambda_{h,k}}{1+\lambda_{h,k}C_h^2} \le \lambda_k \quad (k=1,2,\cdots,n)\,.$$

Recall the eigenvalue problem in V^h

Find $u_h \in V^h$ and $\lambda_h \ge 0$, s.t.,

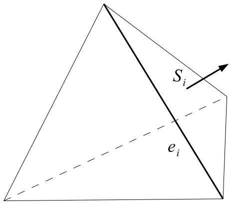
$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in V^h.$$

- Eigenvalues: $\lambda_{h,1} \le \lambda_{h,2} \cdots \le \lambda_{h,n}$.

General Fujino-Morley interpolation operators

Fujino-Morley interpolation in 3D

Extension of Fujino-Morley interpolation in 3D
Let K be a tetrahedron with surfaces S_i, $i = 1, 2, 3, 4$, and edges e_i, $i = 1, \cdots, 6$.



For $u \in H^2(K)$, $\Pi_h u \in P^2(K)$ is determined by

1) $\int_{S_i} \frac{\partial(u-\Pi_h u)}{\partial n} dS = 0, \quad i = 1, 2, 3.$
2) $\int_{e_i} u - \Pi_h u ds = 0, \quad i = 1, \cdots, 6\,.$

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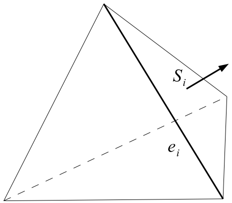

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Property

For any $v \in P^2(K)$,

$$\int_K D^2(u - \Pi_h u) \cdot D^2 v dX = 0$$

Fujino-Morley interpolation in 1D

Extension of Fujino-Morley interpolation in 1D

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For $u \in H^2(I)$, $\Pi_h u \in P^2(I)$ is determined by

1) $(\Pi_h u)^{(1)}(a) = u^{(1)}(a)$, $(\Pi_h u)^{(1)}(b) = u^{(1)}(b)$.
2) $(\Pi_h u)(a) = u(a)$, $(\Pi_h u)(b) = u(b)$.

Fujino-Morley interpolation in 1D

Extension of Fujino-Morley interpolation in 1D

Let I be an interval with two vertices a and b.

For $u \in H^2(I)$, $\Pi_h u \in P^2(I)$ is determined by

1) $(\Pi_h u)^{(1)}(a) = u^{(1)}(a)$, $(\Pi_h u)^{(1)}(b) = u^{(1)}(b)$.
2) $(\Pi_h u)(a) = u(a)$, $(\Pi_h u)(b) = u(b)$.

Property

For any $v \in P^2(I)$,

$$\int_I (u - \Pi_h u)^{(2)} \cdot v^{(2)} ds = 0$$

Upper bound for eigenvalues of Bi-harmonic operator

Upper bound for eigenvalues of Bi-harmonic operator

Variational formulation: I

$V := \{v \in H^2(\Omega) |\ u = \partial u / \partial n = 0$ on $\partial\Omega\}$.

Find $u \in V$ and $\lambda \geq 0$ s.t. $\quad (D^2 u, D^2 v) = \lambda(\nabla u, \nabla v) \quad \forall v \in V$

Variational formulation: II

$D := \{(p_1, p_2) \in H_0^1(\Omega)^2 |$ curl $p = 0$ in $\Omega\}$.

Find $u \in D$ and $\lambda \geq 0$ s.t. $\quad (\nabla p, \nabla q) = \lambda(p, q) \quad \forall q \in D$.

Variational formulation in FEM space

Lagrange FEM space over triangulation $\mathcal{T}^h$

- $L_h^d = \{v_h \in H^1(\Omega) \mid v_h|_K \in P^d(K)$ for $K \in \mathcal{T}^h\}$
- $D_h = \left(L_h^d \times L_h^d\right) \cap D$

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- $D_h = \left(L_h^d \times L_h^d\right) \cap D$

Eigenvalue problem in D_h:

Find $p_h \in D_h$ and $\hat{\lambda}_h \geq 0$ s.t.

$$(\nabla p_h, \nabla q_h) = \hat{\lambda}_h (p_h, q_h) \quad \forall q_h \in D_h.$$

The eigenvalues of the above problem are denoted by $\hat{\lambda}_{h,k}$ $(k = 1, \cdots, n)$ in an increasing order.

Variational formulation in FEM space

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The eigenvalues of the above problem are denoted by $\hat{\lambda}_{h,k}$ $(k = 1, \cdots, n)$ in an increasing order.

- Since $D_h \subset D$, an upper bound for λ_k is given as $\hat{\lambda}_{h,k}$.
- The 4th order differential problem is solved by using H_0^1-conforming FEMs.

Computation results

Domain: a unit isosceles right triangle domain.

Table : Rough eigenvalue bounds

h	λ_1	λ_2	λ_3
1/16	[135.6175, 142.5736]	[194.1698, 213.8170]	[235.5488, 257.0632]
1/32	[138.5584, 140.3105]	[202.6644, 207.5471]	[244.7311, 250.1080]
1/64	[139.3179, –]	[204.8289, –]	[247.0735, –]

Parameters in computing

- Upper bound computation: $D_h = (L_h^2 \times L_h^2) \cap D$, i.e., $d = 2$.
- Constant C_h for P_h: $C_h := 0.24h$.

Another eigenvalue problem for Bi-harmonic operators

- $\Delta^2 u = \lambda u$ in Ω; $u = \partial u / \partial n = 0$ on $\partial\Omega$.

Eigenvalue problem of Bi-harmonic operator

Over bounded domain Ω ($\subset R^2$),

$$\Delta^2 u = \lambda u \text{ in } \Omega; \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

Variational formulation:
$V := \{v \in H^2(\Omega) | \ u = \partial u / \partial n = 0 \text{ on } \partial\Omega\}$.

$$\text{Find } u \in V \text{ and } \lambda \geq 0 \text{ s.t.} \quad (D^2u, D^2v)_\Omega = \lambda(u, v)_\Omega \quad \forall v \in V$$

Let Π_h be Fujino-Morley interpolation operator. We need to give estimation for constant C_h such that

$$\|u - \Pi_h u\|_{0,\Omega} \leq C_h |u - \Pi_h u|_{2,\Omega} .$$

The error estimation for Fujino-Morley interpolation operator Π_h

Error constant $C_0(K)$:

$$\|u - \Pi_h u\|_{0,K} \le C_0(K)|u - \Pi_h u|_{2,K}$$

Constant $C_0(K)$ is characterized by

$$C_0(K) := \sup_{u \in W(K)} \frac{|u|_{0,K}}{|u|_{2,K}} ,$$

where

$$W(K) := \{u \in H^2(K) \mid u(p_i) = 0, \int_{e_i} \frac{\partial u}{\partial n} ds = 0, \ i = 1, 2, 3\} .$$

Upper bound of $C_0(K)$

Introduce an auxiliary space $\tilde{W}(K) \subset W(K)$:

$$\tilde{W}(K) := \{u \in H^2(K) \mid u(p_i) = 0, \ \ i = 1, 2, 3\} .$$

Thus

$$C_0(K) := \sup_{u \in W(K)} \frac{|u|_{0,K}}{|u|_{2,K}} \leq \sup_{u \in \tilde{W}(K)} \frac{|u|_{0,K}}{|u|_{2,K}} .$$

By using Taylor's expansion, we can easily show that for unit isosceles right triangle,

$$C_0(K) \leq \frac{\sqrt{2}}{2} \leq 0.71 .$$

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By using Taylor's expansion, we can easily show that for unit isosceles right triangle,

$$0.0909 \approx C_0(K) \leq \frac{\sqrt{2}}{2} \leq 0.71 \,.$$

Eigenvalue problem of plate

Let $\lambda_{h,k}$ be the approximate eigenvalues from Fujino-Morely FEM. Lower bounds for eigenvalue λ_k,

$$\frac{\lambda_{h,k}}{1+C_h^2\lambda_{h,k}} \le \lambda_k \ .$$

Computation result: Over a unit isosceles right triangle domain

$$\lambda_1 \ge 869.46, \quad \lambda_2 \ge 2444.6 \quad (h = 1/64) \ .$$

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Upper bound: C^1 conforming FEM space is needed.

4. High-precision eigen-bounds from Lehmann-Goerisch's theorem

Challenges in desiring high-precision bounds

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Take the eigenvalue problem of Δ as example.

Katou's bound [Katou, 1949]

Let $\tilde{u} \in D(\Delta)$ be approximate eigenvector, and $\tilde{\lambda} := \|\nabla \tilde{u}\|_\Omega^2 / \|\tilde{u}\|^2$ and $\sigma := \| -\Delta \tilde{u} - \tilde{\lambda}\tilde{u}\| / \|\tilde{u}\|_\Omega$. Suppose that μ and ν satisfy, for certain n,

$$\lambda_{n-1} \le \mu < \tilde{\lambda} < \nu \le \lambda_{n+1}.$$

Thus,

$$\tilde{\lambda} - \frac{\sigma^2}{\nu - \tilde{\lambda}} \le \lambda_n \le \tilde{\lambda} + \frac{\sigma^2}{\tilde{\lambda} - \mu}$$

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- Well-constructed vector $\hat{u}$ can provide high-precision bounds;

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- A priori eigenvalue bounds μ and ν are needed;
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Lehmann-Goerisch's theorem can regarded as extended version of Katou's bound, which can easily deal with clustered eigenvalues.

Lehmann-Goerisch's theorem

Assumptions and notation.

A1 D is a real vector space. M and N are symmetric bilinear forms on D; $M(f,f) > 0$ for all $f \in D$, $f \neq 0$.

A2 There exist sequences $\{\lambda_i\}_{i\in\mathbf{N}}$ and $\{\phi_i\}_{i\in\mathbf{N}}$ such that $\lambda_i \in \mathbf{R}$, $\phi_i \in D$, $M(\phi_i, \phi_k) = \delta_{ik}$ for $i, k \in \mathbf{N}$,

$$M(f,\phi_i) = \lambda_i N(f,\phi_k) \qquad \text{for all } f \in D, i \in \mathbf{N}. \tag{4}$$

$$N(f,f) = \sum_{i=1}^{\infty} (N(f,\phi_i))^2 \qquad \text{for all } f \in D, i \in \mathbf{N}. \tag{5}$$

A3 X is a real vector space; $G : D \to X$ is a linear operator; b is a symmetric bilinear form on X. $b(f,f) \geq 0$ for all $f \in X$ and $b(Gf, Gg) = M(f,g)$ for all $f, g \in D$.

A4 $n \in \mathbf{N}$, $v_i \in D$ for $i = 1, \cdots, n$. $w_i \in X$ satisfies

$$b(Gf, w_i) = N(f, v_i) \text{ for all } f \in D, i = 1, \cdots, n\,; \tag{6}$$

Lehmann-Goerisch's theorem

A5 $\rho \in \mathbf{R}$, $\rho > 0$. Define matrices as

$$A_0 := (M(v_i, v_k))_{i,k=1,\cdots,n}\,, \quad A_1 := (N(v_i, v_k))_{i,k=1,\cdots,n}\,,$$
$$A_2 := (b(w_i, w_k))_{i,k=1,\cdots,n}\,,$$
$$A^L = A_0 - \rho A_1, \quad B^L = A_0 - 2\rho A_1 + \rho^2 A_2;$$

B^L is positive definite. For $i = 1, \cdots, n$, the ith smallest eigenvalue of the eigenvalue problem $A^L z = \mu B^L z$ is denoted by μ_i.

Assertion 1

For all$\mu_i < 0$ $(1 \le i \le n)$, the interval $[\rho - \rho/(1-\mu_i), \rho)$ contains at least i eigenvalues of (4).

Assertion 2

Suppose the ρ in A5 satisifes, $\lambda_m < \rho \le \lambda_{m+1}$. Then, a lower bound of λ_k $(1 \le k \le m)$ is given as

$$\rho - \rho/(1-\mu_k) \le \lambda_{m+1-k}\,.$$

Application of Lehmann-Goerisch's theorem:

- Case of second order problem: $-\Delta u = \lambda u$.

Application of Lehmann-Goerisch's theorem

Eigenvalue problem for the Laplace operator

Ω: polygonal bounded domain; Let $V = \{v \in H^1(\Omega) | v = 0 \text{ on } \partial\Omega\}$.

$$\text{Find } u \in V \text{ and } \lambda \text{ s.t.} \quad (\nabla u, \nabla v) = \lambda(u, v) \quad \forall v \in V\ .$$

The setting for Lehmann-Goerisch's theorem

- $D = V = \{v \in H^1(\Omega) | \int_{e_1} v ds = 0\}$
- $X = (L_2(\Omega))^2$
- $M(u, v) := (\nabla u, \nabla v)$, $N(u, v) := (u, v)$
- $b(w, \tilde{w}) := (w, \tilde{w})$; $M(u, v) = b(\nabla u, \nabla v)$
- $T := \nabla$, $T^* :=$ div,

What does Lehmann-Goerisch's theorem say?

Suppose we have an a priori egienvalue estimation

$$\lambda_2 < \rho \leq \lambda_3 \, .$$

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2. For each $v_i \in V (\subset H^1(\Omega))$, take $w_i \in L_2(\Omega)^2$ such that

$$(w_i, \nabla f)_\Omega = (v_i, f)_\Omega, \quad \forall f \in V \, .$$

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3 Define 2×2 matrices A^L and B^L by using v_1, v_2 and w_1, w_2,

$$A_0 := ((\nabla v_i, \nabla v_k)_\Omega)_{i,k=1,2}\,,\; A_1 := ((v_i, v_k)_\Omega)_{i,k=1,2}\,,\; A_2 := ((w_i, w_k)_\Omega)_{i,k=1,2}\,,$$
$$A^L = A_0 - \rho A_1, \quad B^L = A_0 - 2\rho A_1 + \rho^2 A_2;$$

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4. Let $\mu_1 \le \mu_2$ be the eigenvalues of $A^L x = \mu B^L x$, then

$$\rho - \rho/(1-\mu_2) \le \lambda_1, \quad \rho - \rho/(1-\mu_1) \le \lambda_2 \,.$$

Implementation in finite element spaces

Usually, both v_i and w_i are selected from FEM spaces.

- $L_h^d = \{v_h \in H^1(\Omega) \mid v_h|_K \in P^d(K) \text{ for } K \in \mathcal{T}^h\}$ $(\subset V)$
- RT_h^d : The Raviart-Thomas space of order d $(\subset X)$

What we need is to find a "proper" $w_h \in RT_h^{d+1}$ for given $v_h \in L_h^d$ such that

$$(w_h, \nabla f)_\Omega = (v_h, f)_\Omega, \quad \forall f \in V.$$

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$$(w_h, \nabla f)_\Omega = (v_h, f)_\Omega, \quad \forall f \in V.$$

Saddle point problem to find optimal w_h

Given approximate eigenvector v_h in L_h^d, consider the problem of finding $(w_h, \rho_h) \in RT_h^{d+1} \times X_h^d$ s.t.

$$\begin{cases} (w_h, q_h) + (\rho_h, \text{div } q_h) = 0 & \forall q_h \in RT_h^{d+1} \\ (\text{div } w_h, g_h) + (v_h, g_h) = 0 & \forall g_h \in X_h^d \end{cases} \tag{7}$$

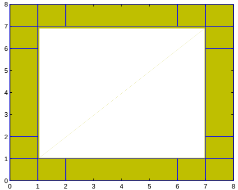
∘ X_h^d: the discontinuous FEM space of order d.

Computation example

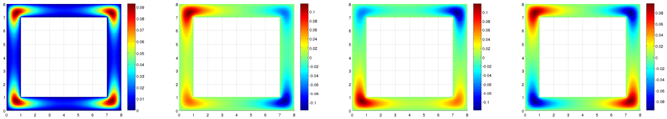
Example : Square-minus-Square domain

Computation parameters:

- Domain: $(0,8)^2 \setminus [1,7]^2$;
- Rough a priori eigenvalue estimation: $\lambda_5 < 35 < \lambda_6$;
- Singular base function used around the re-entrant corners;
- Order of Lagrange FEM space L_h^d: $d = 10$.



λ_i	lower	upper
1	9.1602158	9.1602163
2	9.1700883	9.1700889
3	9.1700883	9.1700889
4	9.1805675	9.1805681



Eigenfunctions corresponding to the leading 4 eigenvalues

Application of Lehmann-Goerisch's theorem:

- Fourth order problem:

$$\Delta\Delta u = -\lambda \Delta u \text{ in } \Omega; \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\,.$$

Variational formulation of the eigenvalue problem

Variational formulation: I

$V := \{v \in H^2(\Omega) |\ u = \partial u / \partial n = 0 \text{ on } \partial\Omega\}.$

$$\text{Find } u \in V \text{ and } \lambda \geq 0 \text{ s.t.} \quad (\Delta u, \Delta v) = \lambda(\nabla u, \nabla v) \quad \forall v \in V$$

Variational formulation: II

$D := \{(p_1, p_2) \in H_0^1(\Omega)^2 |\ \text{curl}\, p = 0 \text{ in } \Omega\}.$

$$\text{Find } u \in D \text{ and } \lambda \geq 0 \text{ s.t.} \quad (\nabla p, \nabla q) = \lambda(p, q) \quad \forall q \in D.$$

Application of Lehmann-Goerisch's theorem

Let $X = L_2(\Omega)^4$. D is defined as before.

- $M(p,q) = (\nabla p, \nabla q))$ for $p,q \in D$
- $N(p,q) = (p,q)$ for $p,q \in D$
- $b(f,g) = (f,g)$ for $f,g \in X$; $M(p,q) = b(\nabla p, \nabla q)$.
- $$T := \begin{pmatrix} \nabla \\ \nabla \end{pmatrix}, \quad T^* := \begin{pmatrix} \text{div} \\ \text{div} \end{pmatrix}.$$

Give an approximation solution $v = (v_1, v_2) \in D_h$, we seek $w \in X$ s.t.

$$b(w, \nabla g) = N(v,g) \text{ for all } g \in D\,. \tag{8}$$

Application of Lehmann-Goerisch's theorem

We further consider the following optimization problem:

$$\min_{w_h \text{ satisfies } (8)} \|w_h\|^2$$

By using the Lagrange multiplier method, we obtain such a saddle point problem.

Saddle point problem

Find $w_h \in \left(RT^{d+1}\right)^2$, $\eta_h \in L_h^{d+1}$, $\rho_h \in \left(DG_h^d\right)^2$ such that

$$(w_h, \tilde{w}_h) + (\text{div} w_h + \text{curl}\, \eta_h, g_h) + (\rho_h, \text{div}\tilde{w}_h + \text{curl}\, \tilde{\eta}_h) = -(v, g_h)$$

for any $\tilde{w}_h \in \left(RT^{d+1}\right)^2$, $\tilde{\eta}_h \in L_h^{d+1}$, $g_h \in \left(DG_h^d\right)^2$.

Computation examples

Buckling plate eigenvalue problem

Buckling plate eigenvalue problem

$$\Delta\Delta u = -\lambda \Delta u, \quad u = \partial u/\partial n = 0 \text{ on } \partial\Omega,$$

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Computing environment

Software/Library:

- FEniCS project with dolfin package. (High-order FEM)
- Boost Interval C++ library, INTLAB interval toolbox of Matlab.

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Computing environment

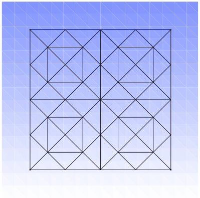
Software/Library:

- FEniCS project with dolfin package. (High-order FEM)
- Boost Interval C++ library, INTLAB interval toolbox of Matlab.

- The computation results are correct up to rounding error.

Buckling plate eigenvalue problem

Unit square domain $\Omega := (0,1)^2$



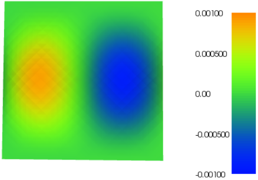


Figure : Left: triangulation for domain; Right: $\partial u/\partial x$

Eigenvalue bounds

- Approximate eigenvalues
  $\lambda_1 \approx 52.3446989, \quad \lambda_2 \approx 92.1244138, \quad \lambda_3 = 92.1244138.$
- Eigenvalue bounds: (64 triangle elements; $d = 6$; $\rho = 85.0$)

$$52.34468 \leq \lambda_1 \leq 52.34470$$

Buckling plate eigenvalue problem

Unit triangle domain T: three vertices $(0,0)$, $(1,0)$, $(1,1)$.

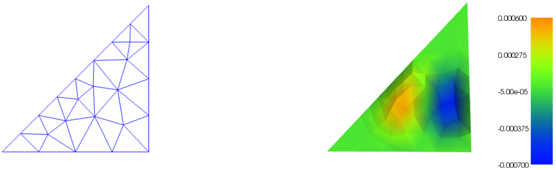


Figure : Left: triangulation for domain; Right: $\partial u/\partial x$

Eigenvalue bounds

- Approximate eigenvalues
  $\lambda_1 \approx 139.574, \quad \lambda_2 \approx 205.554, \quad \lambda_3 \approx 247.864.$
- Eigenvalue bounds: (32 triangle elements; $d = 6$; $\rho = 200.0$)

$$139.57361 \leq \lambda_1 \leq 139.57435$$

Buckling plate eigenvalue problem

L-shaped domain $\Omega := (0,2)^2 \setminus [1,2]^2$

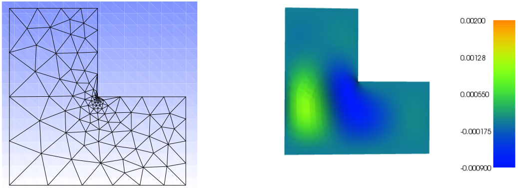


Figure : Left: non-uniform mesh for L-shaped domain; Right: $\partial u / \partial x$

Eigenvalue bounds

- Approximate eigenvalues
  $\lambda_1 \approx 32.14275, \quad \lambda_2 \approx 37.01887, \quad \lambda_3 \approx 41.944099.$
- Eigenvalue bounds: (204 triangle elements; $d = 6$; $\rho = 36.0$)

$$32.09448 \le \lambda_1 \le 32.135839$$

Summary

We proposed a framework to give high-precision eigenvalue bounds: