電磁場有限要素解析において生ずる連立一次方程式とその解法技術

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講演内容

- Part I 電磁場有限要素解析の定式化，解くべき連立一次方程式の性質
- Part II 誤差修正のフレームワーク（EEC, IEC）
- Part III 時間周期電磁場解析の高速化（TP-EEC）
- Part IV 非正則な係数行列を持つ連立一次方程式の求解高速化（Folded preconditioning）

Electromagnetic field analysis

	Static	Low-frequency 50～60Hz (quasi-static)	High-frequency MHz, GHz (wave)
Time domain Linear Non-Linear	FEM BEM	FEM BEM	FDTD FEM BEM
Frequency domain		FEM BEM	FEM BEM

Basic equations

Maxwell equations

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \quad (1)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (2)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad (3)$$

$$\nabla \cdot \boldsymbol{D} = \rho \quad (4)$$

Constitutive equations

$$\boldsymbol{B} = \mu \boldsymbol{H} \quad (5)$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \quad (6)$$

Ohm's law

$$\boldsymbol{J} = \sigma \boldsymbol{E} \quad (7)$$

H: Magnetic field
E: Electric field
B: Magnetic flux density
D: Electric flux density
ρ: Electric charge density
J: Current density
μ: Permeability
ε: Permittivity
σ: Conductivity

High frequency electromagnetic field analyses (brief introduction)

- Mostly linear analyses
- Time-domain analyses
  - The most popular method is FDTD (finite differential time domain) method.
- Frequency domain analyses
  - FEM (finite element method) is often used.
  - Basic equation

$$\nabla \times \frac{1}{\mu}(\nabla \times \boldsymbol{E}) - \omega^2 \left(\varepsilon + \frac{\sigma}{\mathrm{i}\omega}\right)\boldsymbol{E} = \mathrm{i}\omega \boldsymbol{J}_0$$

$\boldsymbol{J}_0$: external current density

  - The linear system derived from FEM is ill-conditioned

Low frequency electromagnetic field analyses (eddy current analysis)

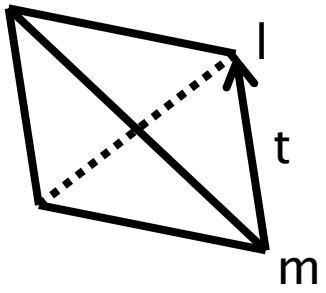
- The displacement current is ignored.
- Non-linear problems are solved in practical simulations for motors, transformers, etc.
- Potential formulation is often used.
  - $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is the vector potential.
  - $\boldsymbol{J} = \boldsymbol{J_c} + \boldsymbol{J_0}$, where $\boldsymbol{J_c}$ is the eddy current in the conductive region.
  - $\boldsymbol{J_c} = \sigma \boldsymbol{E}, \boldsymbol{E} = -\frac{\partial(\boldsymbol{A}+\mathbf{grad}\varphi)}{\partial t}$, where φ is the electric scalar potential.
- Gauge condition should be payed attention to.
  - For fixed $\boldsymbol{B}$, $\boldsymbol{A}$ is not uniquely determined. (in non-conductive region)
  - For fixed $\boldsymbol{B}$ and $\boldsymbol{E}$, a set of ($\boldsymbol{A}$, φ) is not uniquely determined. (in conductive region)

Basic equations (A-φ formulation)

- $\nabla \times \frac{1}{\mu}(\nabla \times \boldsymbol{A}) + \sigma \frac{\partial(\boldsymbol{A}+\mathbf{grad}\varphi)}{\partial \mathrm{t}} = \boldsymbol{J}_0$ (8)
- $\nabla \cdot (\sigma \frac{\partial(\boldsymbol{A}+\mathbf{grad}\varphi)}{\partial \mathrm{t}}) = 0$ (9)
- $\nabla \cdot \boldsymbol{J}_0 = 0$ must be satisfied.
- Without a gauge condition, multiple solutions for ($\boldsymbol{A}$, φ) exist.
- Gauge condition
  - $\nabla \cdot \boldsymbol{A} = 0$ (Coulomb gauge)
  - $\varphi = 0$ (in conductive region)
  - Tree co-tree gauge (after discretization) (in non-conductive region)

Discretization (edge elements)

- The edge element is generally used for the magnetic vector potential.
- The unknown is set on each edge of an mesh.
  - The unknown is given by the integral of the field value along the edge.
- The vector basis function $\boldsymbol{N}$ is used.



First order tetrahedral element

$$\boldsymbol{N}_t = N_m \nabla N_l - N_l \nabla N_m$$

N is the fist order scalar basis function.

Discretization of A: $\boldsymbol{A} = \sum_i \boldsymbol{N}_i A_i$

Linear system

- A linear system of equations is derived from the weak formulation of the basic equations and the discretization. Galerkin method is used (test function = basis function).

$$\begin{cases} [C]\boldsymbol{x}_A + \dfrac{d}{dt}\left([M_\sigma]\boldsymbol{x}_A + [K_{A\varphi}]\boldsymbol{x}_\varphi\right) = \boldsymbol{b}_J \\ \dfrac{\partial}{\partial t}\left([K_{\varphi A}]\boldsymbol{x}_A + [K_{\varphi\varphi}]\boldsymbol{x}_\varphi\right) = 0 \end{cases} \quad (10)$$

$$[C]_{ij} = \sum_e \int_\Omega \frac{1}{\mu}(\nabla \times \boldsymbol{N}_i)\cdot(\nabla \times \boldsymbol{N}_j)dv$$

$$[M_\sigma]_{ij} = \sum_e \int_\Omega \sigma\ \boldsymbol{N}_i \cdot \boldsymbol{N}_j dv$$

$$[K_{A\varphi}]_{ij} = \sum_e \int_\Omega \sigma\ \boldsymbol{N}_i \cdot \mathrm{grad}\, N_j\, dv$$

$$[K_{\varphi A}]_{ij} = \sum_e \int_\Omega \sigma\ \mathrm{grad}\, N_i \cdot \boldsymbol{N}_j\, dv$$

$$[K_{\varphi\varphi}]_{ij} = \sum_e \int_\Omega \sigma\ \mathrm{grad}\, N_i \cdot \mathrm{grad}\, N_j\, dv$$

$$[\boldsymbol{b}_J]_i = \sum_e \int_\Omega \sigma\ \boldsymbol{N}_i \cdot \boldsymbol{J}_0\, dv$$

Linear system

- Frequency domain analysis
  - $\frac{d}{dt} \rightarrow \mathrm{i}\omega$
  - The coefficient matrix results in complex symmetric.
  - IC-COCG method is often used to solve the linear system.
- Time domain analysis
  - When the backward time difference scheme is used, $\frac{dx_A}{dt} \rightarrow \frac{x_A - x_A^{old}}{\Delta t}$, $\frac{dx_\varphi}{dt} \rightarrow \frac{x_\varphi - x_\varphi^{old}}{\Delta t}$.
  - The resulting linear system:

$$\begin{bmatrix} K_{AA} & K_{A\varphi} \\ K_{\varphi A} & K_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_A \\ \boldsymbol{x}_\varphi \end{bmatrix} = \boldsymbol{b} \quad (11)$$

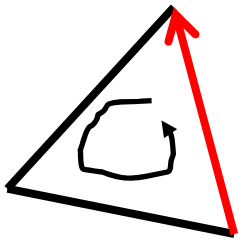
$$[K_{AA}] = [C] + \frac{1}{\Delta t}[M_\sigma]$$

The right-hand side vector $\boldsymbol{b}$ is determined by the external current density and $\boldsymbol{x}_A$ and $\boldsymbol{x}_\varphi$ in the previous time step.

Linear solver

- The coefficient matrix of the linear system (11) is symmetric and semi-positive definite.
- The standard solver is ICCG method.
  - The application of AMG to the edge element analysis is not easy.
  - Other preconditioning techniques and iterative solvers are used in some cases, for example, implementation on GPU.
- Application of the gauge condition
  - Applying $\varphi=0$ to the conductive region, we get A-formulation. The resulting linear system: $K_{AA}\boldsymbol{x}'_A = \boldsymbol{b}'$.
  - K_{AA} is still singular. Applying the tree co-tree gauge condition to the non-conductive region, we get a non-singular coefficient matrix.

Tree co-tree gauge condition



Co-tree edge

- The rotation of the field in the element can be determined by the valuable on the co-tree edge even if the valuable on the tree edges is fixed at an arbitrary value.

Convergence of ICCG method

- When the $\varphi=0$ and the tree co-tree gauge conditions are used, the number of unknowns are reduced.
- However, it is known that the convergence of the ICCG method is significantly deteriorated.
  - Usually, the computational time is increased when these gauge conditions are explicitly enforced.
- Why?
  - References
    - Prof. Honma, Prof. Igarashi, and Prof. Kawaguchi's text book in Japanese: ISBN4-267-71641-9
    - K. Fujiwara, T. Nakata, and H. Ohashi, "Improvement of convergence characteristic of ICCG method for the A -φ method using edge elements," *IEEE Trans. Magn.*, vol. 32, pp. 804-807, May 1996.
    - H. Igarashi and T. Honma, "On convergence of ICCG applied to finite element equation for quasi-static fields," *IEEE Trans. Magn.*, vol. 38, no. 2, pp. 565-568, Mar. 2002.

What is the key?

- The large null space of discrete curl operator
  - The coefficient matrix arising in the A-formulation:

    $[K_{AA}] = [C] + \frac{1}{\Delta t}[M_\sigma]$

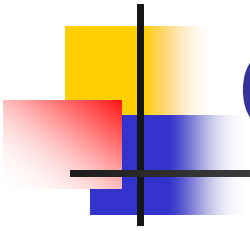
    - The matrix C has large null space, which is sifted by the positive-definite matrix $\frac{1}{\Delta t}M_\sigma$. When the conductivity is small, the shift generates small positive eigenvalues, which causes worse convergence.
    - The error belonging to the null space of the curl operator may converge slowly (discussion in part II).
  - In the linear system based on the A-$\boldsymbol{\varphi}$ formulation, the removal of a redundant unknown causes a smaller eigenvalue in the reduced dimensional coefficient matrix.

Effect of the null space of the curl operator

- Difficulty in AMG
  - The null space matching between fine and coarse grids
- Difficulty in high-frequency analyses
  - Indefinite matrix
  - Influence upon the condition number

Part II
Explicit / Implicit Error Correction Framework

- Collaborators
  - Takeshi Mifune and Masaaki Shimasaki (Kyoto Univ.)
- References
  - Takeshi Iwashita, Takeshi Mifune and Masaaki Shimasaki, “Similarities Between Implicit Correction Multigrid Method and A-phi Formulation in Electromagnetic Field Analysis”, IEEE Transaction on Magnetics, Vol. 44, No. 6, (2008), pp. 946-949.
  - Takeshi Mifune, Soichi Moriguchi, Takeshi Iwashita and Masaaki Shimasaki, “Convergence Acceleration of Iterative Solvers for the Finite Element Analysis Using the Implicit and Explicit Error Correction Methods”, IEEE Transaction on Magnetics, Vol. 45, No. 3, (2009), pp. 1104-1107.
  - Takeshi Iwashita, Takeshi Mifune, Soichi Moriguchi and Masaaki Shimasaki, “Physical Meaning of the Advantage of A-phi Method in Convergence”, IEEE Transaction on Magnetics, Vol. 45, No. 3, (2009), pp. 1424-1427.



Outline

- We focus on some error correction methods in iterative solver.
- We newly define Explicit error correction method for a class for some conventional techniques.
- We propose Implicit error correction method.
  - Then, we establish Explicit / Implicit error correction framework
- Introduction of two techniques described in the proposed framework
- Numerical results confirm the effectiveness of implicit error correction method.

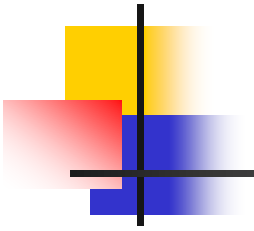
Explicit error correction method
(subspace correction method)

- Definition
  - $Ax=b$ (1) : A linear system to be solved
    - A: n by n matrix, x, b: n-dimensional vector
  - If an error correction method is given by the following two steps, we call it one of "explicit error correction methods".
  - Step 1: Determination of quantity of error correction
  - Solve or use $$Cy = d(\tilde{x})$$ (2)
    - C : m by m matrix, d : a function of the approximate solution $\tilde{x}$.

Explicit error correction method

- Step 2: Error correction (update of $\tilde{x}$)
  - $$\tilde{x}^{new} \leftarrow \tilde{x} + B\tilde{y} \quad (3)$$
  - $\tilde{y}$: the approximation or exact y
  - B : n by m matrix
  - The dimension m is (sufficiently) smaller than n in practical use.

Characteristics of explicit error correction method



- Special treatment for errors that converge slow.
- Use of a different matrix from a original coefficient matrix
- Famous examples
  - Hiptmair's hybrid smoother
  - Coarse grid correction in a multigrid method

Implicit error correction method

- We propose a new technique "implicit error correction method".
- The idea comes from solving the original linear system (1) and the linear system for error correction (2) simultaneously.
- Derivation of the proposed method
  - Paying special attention to (3), the solution vector x is replaced by $\hat{x} + B\hat{y}$ in (1).

    Then, we get $A\hat{x} + AB\hat{y} = b$ (4).

Implicit error correction method

- Next, we consider (2). In most explicit error correction methods, d is given by a restricted residual equation of (1):

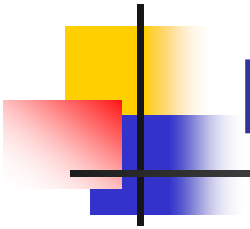
$$d = D(b - A\tilde{x}) .$$

- Thus, we obtain $\boxed{DA\tilde{x} + Cy = Db}$.
- By rewriting vectors $\tilde{x}$ and y by $\hat{x}$ and $\hat{y}$, respectively, we finally obtain

$$\begin{pmatrix} A & AB \\ DA & C \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} b \\ Db \end{pmatrix} \quad (6).$$

Implicit error correction method

- The above linear system (6) is the formulation based on the implicit error correction method.
- The linear system (6) is solved by a preconditioned iterative solver in the proposed method.
- It is expected that the iterative solution process produces the effect of the error correction implicitly. In other words, the linear system (6) has an improved condition of the coefficient matrix over that of the original linear system (1).



Electromagnetic field simulation

- The advantage in convergence in the A-phi method has not been explained in physical meaning.
- We show that the A-phi method can be regarded as a implicit error correction method corresponding to Hiptmair's hybrid smoother.
- Numerical tests show the A-phi method has similar error correction effect as the hybrid smoother.

Two formulations in electromagnetic field simulation

- A-method

$$\nabla \times (\nu \nabla \times A_m) + \sigma \frac{\partial A_m}{\partial t} = J_0 \qquad (7)$$

- A-phi method

$$\nabla \times (\nu \nabla \times A'_m) + \sigma \frac{\partial (A'_m + \mathrm{grad}\phi)}{\partial t} = J_0 \qquad (8)$$

$$\mathrm{div}(\sigma \frac{\partial (A'_m + \mathrm{grad}\phi)}{\partial t}) = 0 \qquad (9)$$

A_m, A_m' : the magnetic vector potential,

ν: the magnetic reluctivity, σ: the electrical conductivity,

t: time, J_0: the exciting current, ϕ, the electric scalar potential

$\nabla \cdot J_0 = 0$ is satisfied.

Linear system based on A-formulation

Application of the finite edge element discretization and a backward time difference method to (7) results in

$$\left[K_A\right]=\left[C_u{}^T M_\nu C_u+\frac{M_\sigma}{\Delta t}\right]\left\{x_A\right\}=\left\{b_A\right\} \qquad (10)$$

where Δt: the length of the time step, x_A: the unknown vector for magnetic vector potential, b_A is determined by the previous value of the magnetic vector potential and the exiting current.

C_u: the discrete curl operator

$$M_\nu=\int_\Omega \nu w_i^f\cdot w_j^f dv \qquad M_\sigma=\int_\Omega \sigma w_i^e\cdot w_j^e dv \qquad (11)$$

w_f: face-element basis function, w_e: edge-element basis function

We consider the application of Hiptmair's hybrid smoother to this linear system.

Relationship between A-phi method and Hiptmair's hybrid smoother

- When the **hybrid smoother** is applied to the linear system (A-method) (10), its error correction procedure can be written in a form of explicit error correction method as follows:

$$C = (G^T M_\sigma G)/\Delta t$$
$$d(\tilde{x}) = G^T (b_A - K_A \tilde{x}_A)$$
$$B = G$$

in (2) and (3)

G : discrete gradient operator, $C_u G = 0$ is satisfied.

Corresponding implicit error correction method…

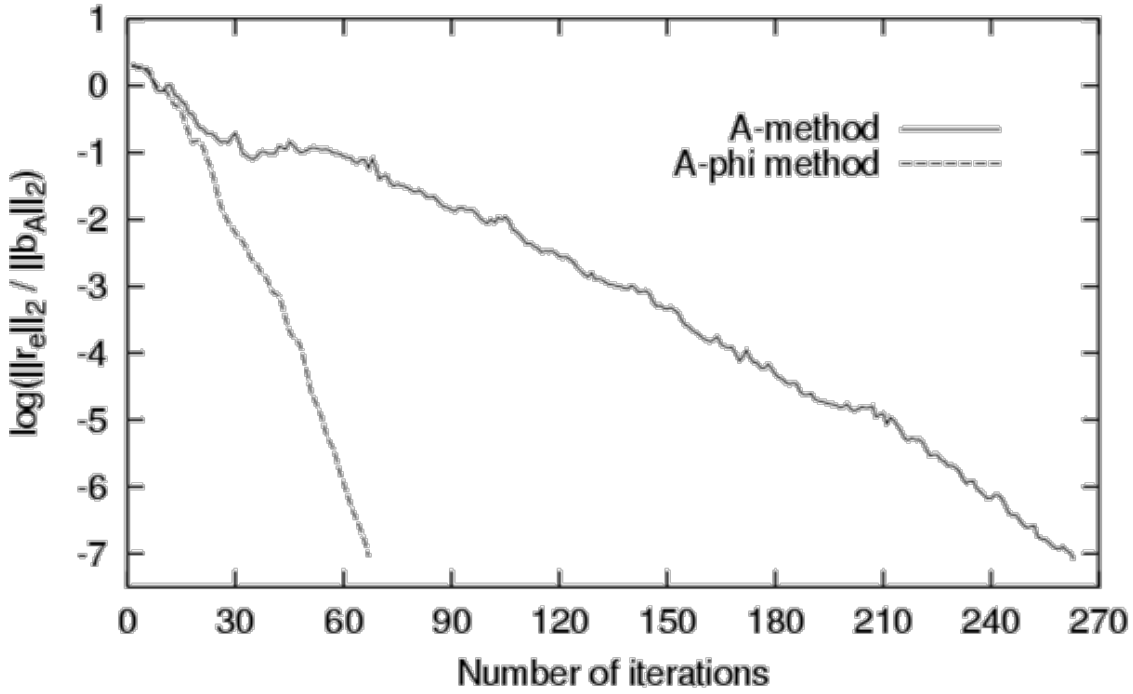
$$\begin{pmatrix} K_A & \frac{1}{\Delta t} M_\sigma G \\ \frac{1}{\Delta t} G^T M_\sigma & \frac{1}{\Delta t} G^T M_\sigma G \end{pmatrix} \begin{pmatrix} \hat{x}_A \\ \hat{y}_A \end{pmatrix} = \begin{pmatrix} b_A \\ G^T b_A \end{pmatrix}$$

This linear system coincides with the linear system arising on the A-phi formulation.

- The A-phi method can be regarded as a implicit error correction method corresponding to Hiptmair's hybrid smoother.
- Thus, it is expected that the A-phi method has a similar effect to that of the hybrid smoother for the error of the kernel of the discrete curl operator.
- Whereas the hybrid smoother corrects these errors explicitly, the A-phi method introduces unknown variables for the electrical scalar potential.

Numerical tests (Effectiveness of A-phi method)

- Test model: TEAM workshop problem 10
- Whitney elements, # DOFs 5968
- $\Delta t = 10^{-3}$(sec)



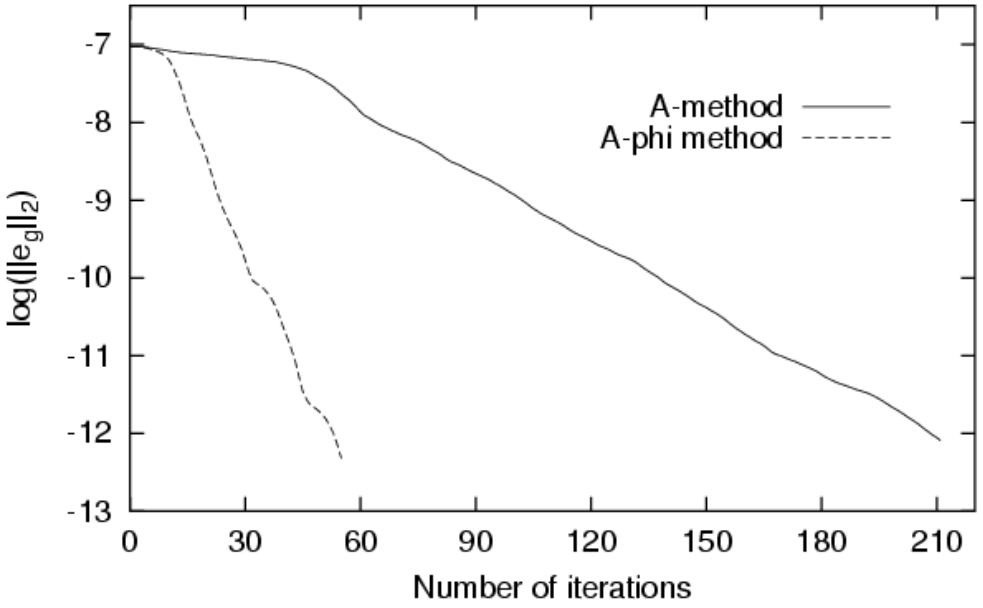
Comparison behavior (relative residual norm)

Well-known property

This phenomena has been explained by the condition number of coefficient matrices.

Numerical tests (Effectiveness of A-phi method)

- We focus on the error of the kernel of the discrete curl operator *K*(curl) in conductive region.
- This error is explicitly corrected in Hiptmair's hybrid smoother.

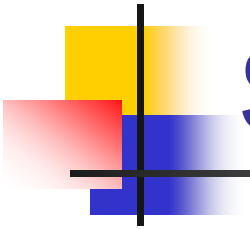


• The error of *K*(curl) in conductive region is efficiently removed in the A-phi method.

• Numerical tests confirm that the A-phi method has a similar effect as that of the hybrid smoother for the error of *K*(curl) in conductive region.

Numerical tests (Effectiveness of A-phi method)

- The error of *K*(curl) in conductive region is efficiently removed in the A-phi method.
- The slow convergence of the A-method is caused by the error of *K*(curl) in conductive region.
- Numerical tests confirm that the A-phi method has a similar effect as that of the hybrid smoother for the error of *K*(curl) in conductive region.
- **We give the first physical explanation of the advantage of A-phi method.**



Summary of part II

- We propose an implicit error correction method that corresponds to the explicit error correction methods such as Hiptmair's hybrid smoother.
- It is shown that the A-phi method can be regarded as an implicit error correction version of the hybrid smoother.
- Numerical tests show that the A-phi method has a similar effect on the correction of the error of the kernel of the discrete curl operator, which results in an advantage in convergence.

Part III : TP-EEC Method

- Main contributors
  - Yasuhito Takahashi (Doshisha Univ.), Tadashi Tokumasu (Toshiba Co. Ltd.)
- Reference
  - Y. Takahashi, T. Tokumasu, A. Kameari, H. Kaimori, M. Fujita, T. Iwashita, and S. Wakao, “Convergence Acceleration of Time-Periodic Electromagnetic Field Analysis by the Singularity Decomposition-Explicit Error Correction Method,” IEEE Transactions on Magnetics, Vol. 46, No. 8, 2010.

Target problem

- Motors and transformers (practical simulations)
- Low frequency field problem (50-60Hz)
  - Eddy current problem
- Non-linear problem
  - Non-linear magnetic property
- Calculation for time-periodic steady state solution

Basic equations

- A-phi formulation

$$\begin{cases} \nabla \times (\nu \nabla \times A) + \sigma \dfrac{\partial (A + \mathrm{grad}\,\phi)}{\partial t} = J_0 \\ \mathrm{div}\left(\sigma \dfrac{\partial (A + \mathrm{grad}\,\phi)}{\partial t}\right) = 0 \quad \text{in conductive region} \end{cases}$$

A : the magnetic vector potential, ν: the magnetic reluctivity, σ: the electrical conductivity, t: time, J_0: the exiting current, ϕ, the electric scalar potential

PDE problem -> FEM

ν has non-linear property: $\nu = \nu$(rot A) -> Newton Raphson method

Transient analysis should be performed.

Treatment of the time derivative -> (1) Step by step method
(2) TPFEM

Conventional method

Transient analysis

$$S(\boldsymbol{x}) + C\frac{\partial}{\partial t}\boldsymbol{x} = \boldsymbol{f}$$

A system of nonlinear equations derived from the finite element discretization

Adopting the θ method for the time integration scheme

$$\tilde{T}(\boldsymbol{x}_i) - \tilde{C}(\boldsymbol{x}_{i-1}) = \tilde{\boldsymbol{f}}_i$$

$$\begin{cases} \tilde{T}(\boldsymbol{x}_i) = \theta S(\boldsymbol{x}_i) + \dfrac{C}{\Delta t}\boldsymbol{x}_i \\ \tilde{C}(\boldsymbol{x}_i) = -(1-\theta) S(\boldsymbol{x}_i) + \dfrac{C}{\Delta t}\boldsymbol{x}_i \\ \tilde{\boldsymbol{f}}_i = \theta\, \boldsymbol{f}_i + (1-\theta)\, \boldsymbol{f}_{i-1} \end{cases}$$

(The subscripts indicate time step.)

+

Time periodic condition

- Ordinary time-periodic condition: $x(t+T) = x(t)$
- Half time-periodic condition: $x(t+T/2) = -x(t)$

T is the period.

Step by step method

In the step-by-step method, a series of the following non-linear equations are repeatedly solved until the steady state solution is obtained.

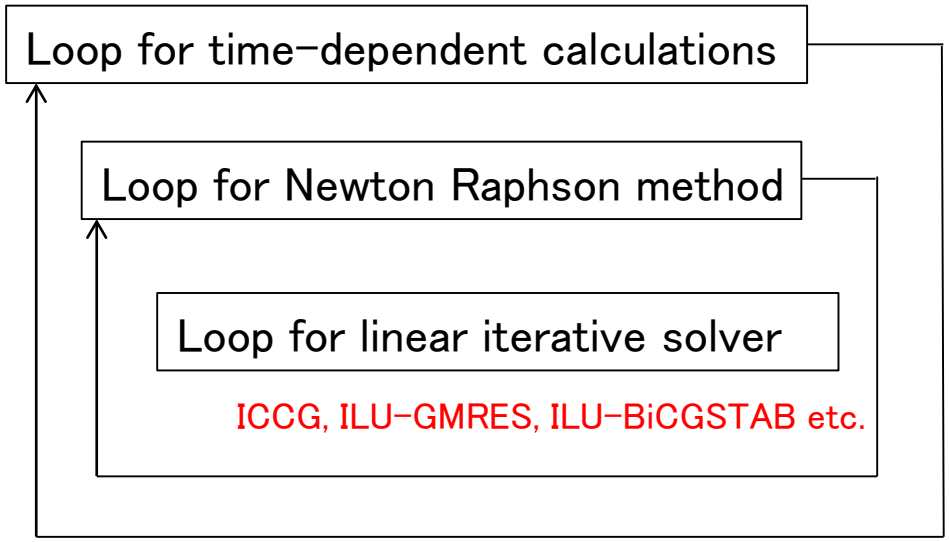
$$\mathbf{G}_i = \tilde{\mathbf{f}}_i - \tilde{T}\left(\mathbf{x}_i\right) + \tilde{C}\left(\mathbf{x}_{i-1}\right) = 0 \qquad (i = 1, 2, \cdots, n_t)$$

$$\boldsymbol{x}_0 = \begin{cases} 0 & \text{Initial guess (only used in the first time step)} \\ \boldsymbol{x}_{nt} & \text{Ordinary time-periodic condition case} \\ -\boldsymbol{x}_{nt} & \text{Half time-periodic condition case} \end{cases}$$

Each non-linear equation has *n* unknowns, where *n* is the number of unknowns of the analyzed model (discrete vector and scalar potentials).

Overview of the simulation code based on the step-by-step method

- The simulation code consists of three nested loop (for time integration, Newton method, linear solver).



Computational cost: $C \times n \times N_{ite} \times N_r \times N_T$

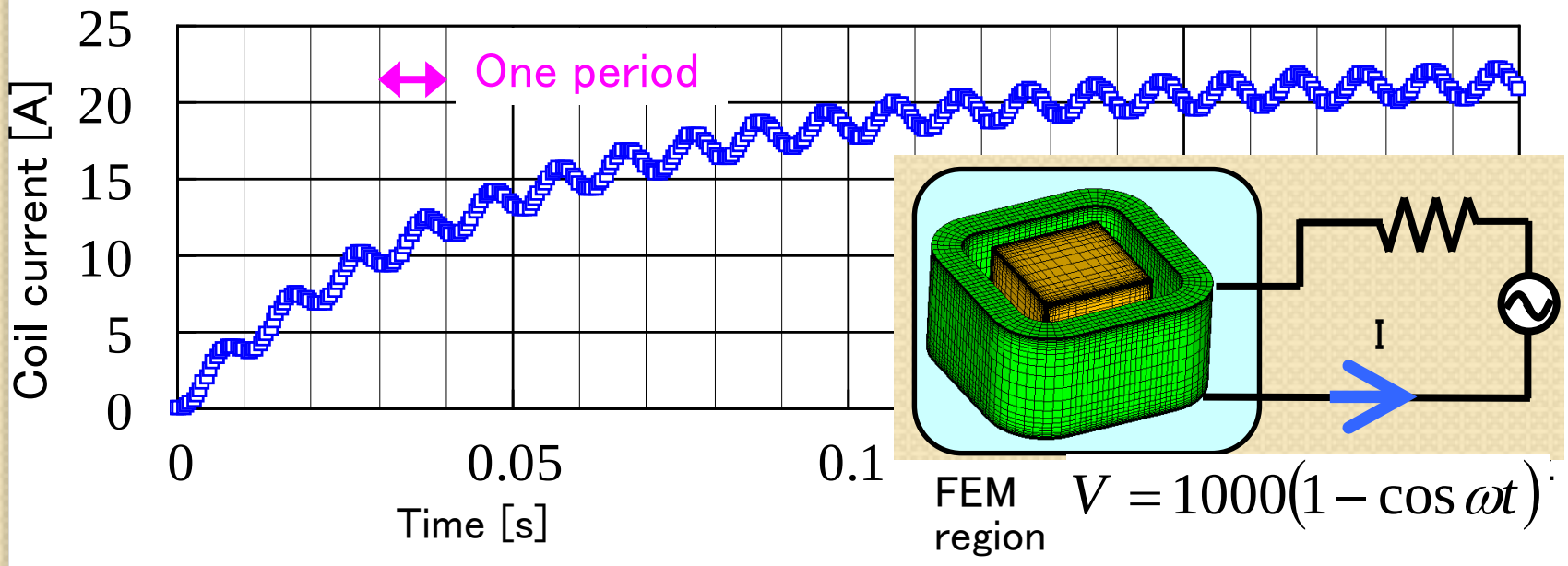
C : a constant value determined by the linear iterative solver
N_{ite} : the average number of iterations of iterative solver
N_r : the average number of non linear iterations
N_T: the number of total time steps

A problem of the step by step method

The transient analysis is a kind of initial value problems of ODE.

If the time constant of the problem is very large, many time steps are needed to obtain the steady state solution.

In practical simulations, the analyzed model involves an external circuit, which often leads a large time constant.



Time-periodic finite-element method (TPFEM)

- All linear systems are combined with the time periodic condition and are solved simultaneously.

$$\begin{bmatrix} \tilde{T}(\boldsymbol{x}_1) \mp \tilde{C}(\boldsymbol{x}_n) \\ \tilde{T}(\boldsymbol{x}_2) - \tilde{C}(\boldsymbol{x}_1) \\ \tilde{T}(\boldsymbol{x}_3) - \tilde{C}(\boldsymbol{x}_2) \\ \vdots \\ \tilde{T}(\boldsymbol{x}_{n_t}) - \tilde{C}(\boldsymbol{x}_{n_t-1}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \\ \vdots \\ \boldsymbol{f}_{n_t} \end{bmatrix}$$

Solve this large system of non-linear equations by means of Newton Raphson method.

However, the convergence of the Newton method is not good, and thus TPFEM is usually inferior to the step by step method in the computational time.

But, TPFEM has an advantage in parallel computation.

Main idea

- Acceleration of convergence in the transient analysis by using EEC (Explicit Error Correction) (Subspace correction) method

The problem: linear system of equations Kx=b

The approximation vector $\tilde{x}$ is updated as follows:

$$\boldsymbol{r} = \boldsymbol{b} - K\tilde{\boldsymbol{x}}$$

$B^T KB\boldsymbol{y} = B^T \boldsymbol{r}$ is (approximately) solved.

$$\tilde{\boldsymbol{x}}_{new} = \tilde{\boldsymbol{x}} + B\tilde{\boldsymbol{y}}$$

EEC method for non-linear equation

$K(\boldsymbol{x}) = \boldsymbol{b}$: the nonlinear equation to solve

$K(\tilde{\boldsymbol{x}} + \boldsymbol{e}) = \boldsymbol{b}$ ⇩ Linearization

Equation for the error

$$K(\tilde{\boldsymbol{x}}) + \frac{\partial K(\tilde{\boldsymbol{x}})}{\partial \boldsymbol{x}} \boldsymbol{e} = \boldsymbol{b} \quad \Rightarrow \quad \frac{\partial K(\tilde{\boldsymbol{x}})}{\partial \boldsymbol{x}} \boldsymbol{e} = \boldsymbol{b} - K(\tilde{\boldsymbol{x}})$$

$$B^T \frac{\partial K(\tilde{\boldsymbol{x}})}{\partial \boldsymbol{x}} B\boldsymbol{e}_H = B^T (\boldsymbol{b} - K(\tilde{\boldsymbol{x}}))$$

Equation for error correction (lower-rank)

$$\tilde{\boldsymbol{x}}^{new} \leftarrow \tilde{\boldsymbol{x}} + B\boldsymbol{e}_H$$

Equation for error correction (lower-lank)

EEC Procedure

What kind of errors are slow in convergence?

- Tokumasu and Takahashi's idea
  - The DC component of the error can have a low convergence rate.
  - How is it extracted ?
    - Use TPFEM formulation (for K(x)=b)
    - The DC component of the error has the identical value of potentials in all time steps.

      The DC error component is written as

      $(e_1, e_2, \cdots, e_n, e_1, e_2, \cdots, e_n, \cdots\cdots, e_1, e_2, \cdots, e_n)^{\mathrm{T}}$

      in TPFEM formulation.

      It is rewritten as $\underbrace{(\boldsymbol{I}\,\boldsymbol{I}\cdots\boldsymbol{I})^{T}}_{B}\underbrace{(e_1, e_2, \cdots, e_n)^{\mathrm{T}}}_{\boldsymbol{e}_{\mathrm{H}}}$

      $\boldsymbol{I}$: n x n unit matrix

TP-EEC method

(1) Perform the step by step method for *nt* time steps

(2) Perform the following error correction for the current approximation vector $\boldsymbol{x}_{nt}$

Solve $B^T \dfrac{\partial K(\tilde{\boldsymbol{x}})}{\partial \boldsymbol{x}} B\boldsymbol{e}_H = B^T(\boldsymbol{b} - K(\tilde{\boldsymbol{x}}))$

$\tilde{\boldsymbol{x}}^T = (\boldsymbol{x}_0^T, \boldsymbol{x}_1^T, \ldots, \boldsymbol{x}_{nt}^T)$

$B = (I, I, \ldots, I)^T$ (*n* x *nt*) by *n* matrix

K : the coefficient matrix in TPFEM

Update $\boldsymbol{x}_{nt}$ to $\boldsymbol{x}_{nt} + B\boldsymbol{e}_H$

(3) If the convergence condition is not satisfied, return to (1) with $\boldsymbol{x}_0 = \pm \boldsymbol{x}_{nt}$.

Simplified TP-EEC

- The technique is available only for a half time periodic problem.

$$\left(\sum_{i=1}^{nt}\left(\tilde{T}_i-\tilde{C}_i\right)+2\tilde{C}_{nt}\right)\boldsymbol{e}_H=-\tilde{C}(\boldsymbol{x}_0)-\tilde{C}(\boldsymbol{x}_{nt})$$

The equation for error correction

- Assume that $\boldsymbol{e}_H=\alpha\boldsymbol{q}=\alpha(-\boldsymbol{x}_0-\boldsymbol{x}_{nt})$
- Adopt the backward Euler method (θ=1)
- Multiply both sides by $\boldsymbol{q}^{\mathrm{T}}$

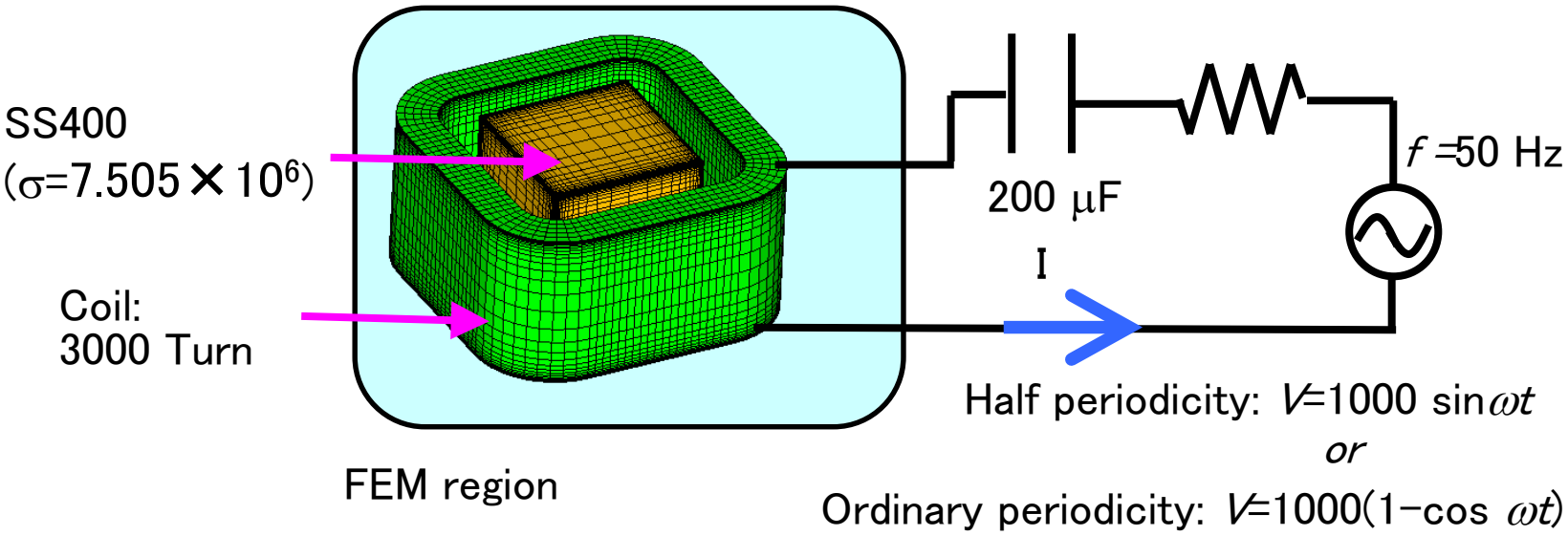
$$\alpha=\frac{1}{\left(\Delta t\,\boldsymbol{q}^T\left(\sum_{i=1}^{n}S_i\right)\boldsymbol{q}\Big/\boldsymbol{q}^T C\,\boldsymbol{q}+2\right)}$$

When the influence of the matrix C is larger than S_i, which corresponds to the case that the time constant is large, α = 1/2

$$\boldsymbol{e}_H=\frac{1}{2}(-\boldsymbol{x}_0-\boldsymbol{x}_{nt})$$

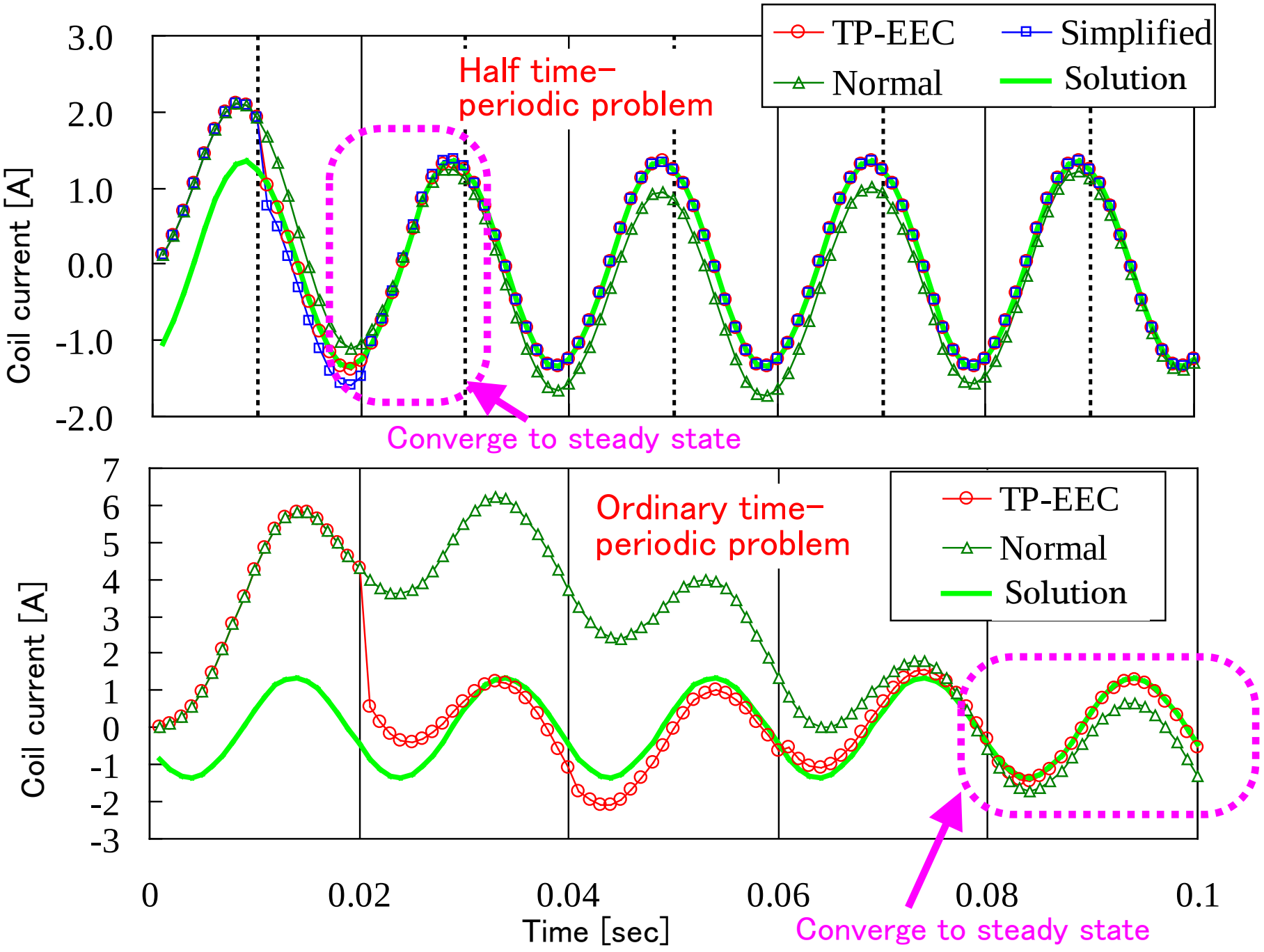
The simplified TP-EEC method, which is unnecessary to solve the linear system for the error.

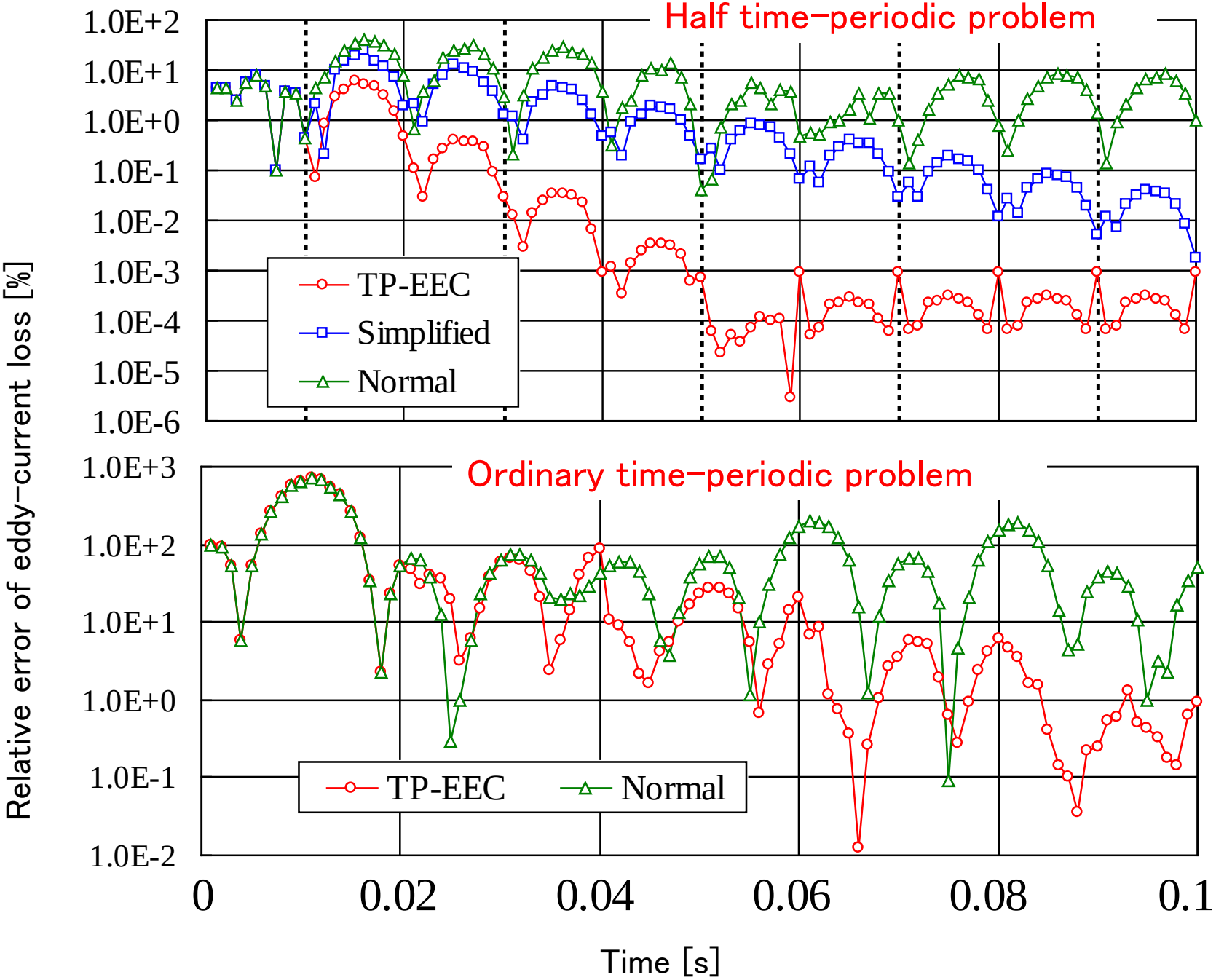
Numerical tests (Iron core model)



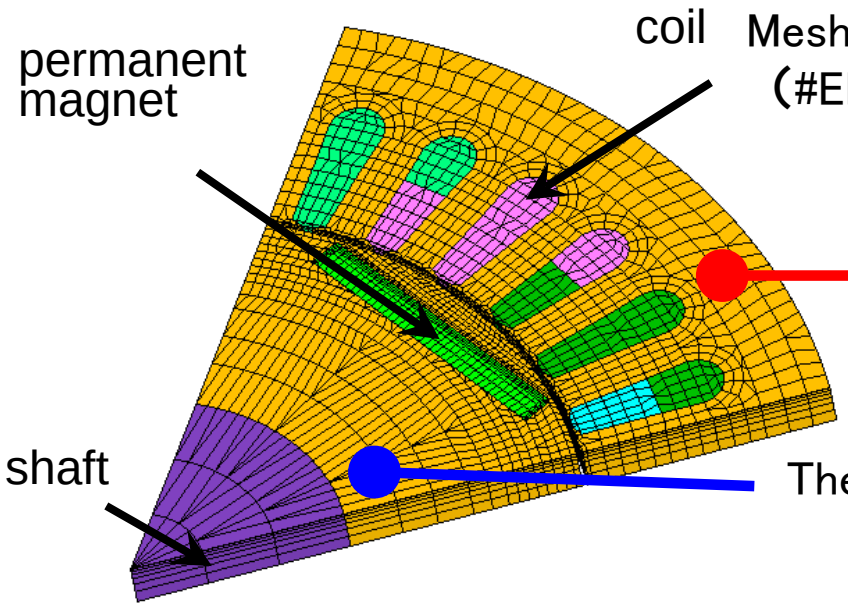
Methods	Elapsed time (s)	
	Half periodic problem	Periodic problem
Normal step-by-step	3360.4 (207)	6848.7 (432)
TP-EEC method	648.7 (39)	1999.3 (121)
Simplified method	1101.8 (68)	

Numbers in parenthesis indicate the number of time steps required for obtaining steady state solutions in one period


TP-EEC
Simplified
Normal
Solution
Half time–
periodic problem
Converge to steady state
Coil current [A]
3.0
2.0
1.0
0.0
-1.0
-2.0
Ordinary time–
periodic problem
TP-EEC
Normal
Solution
Coil current [A]
7
6
5
4
3
2
1
0
-1
-2
-3
0
0.02
0.04
0.06
0.08
0.1
Time [sec]
Converge to steady state


Half time-periodic problem
1.0E+2
1.0E+1
1.0E+0
1.0E-1
1.0E-2
1.0E-3
1.0E-4
1.0E-5
1.0E-6
TP-EEC
Simplified
Normal
Ordinary time-periodic problem
1.0E+3
1.0E+2
1.0E+1
1.0E+0
1.0E-1
1.0E-2
TP-EEC
Normal
Relative error of eddy-current loss [%]
0
0.02
0.04
0.06
0.08
0.1
Time [s]

Numerical test (IPM motor)



Mesh of IPM motor (1/6 model)
(#Elements：17,556, #Unknowns：49,798)

The stator is satisfied with half time-periodicity

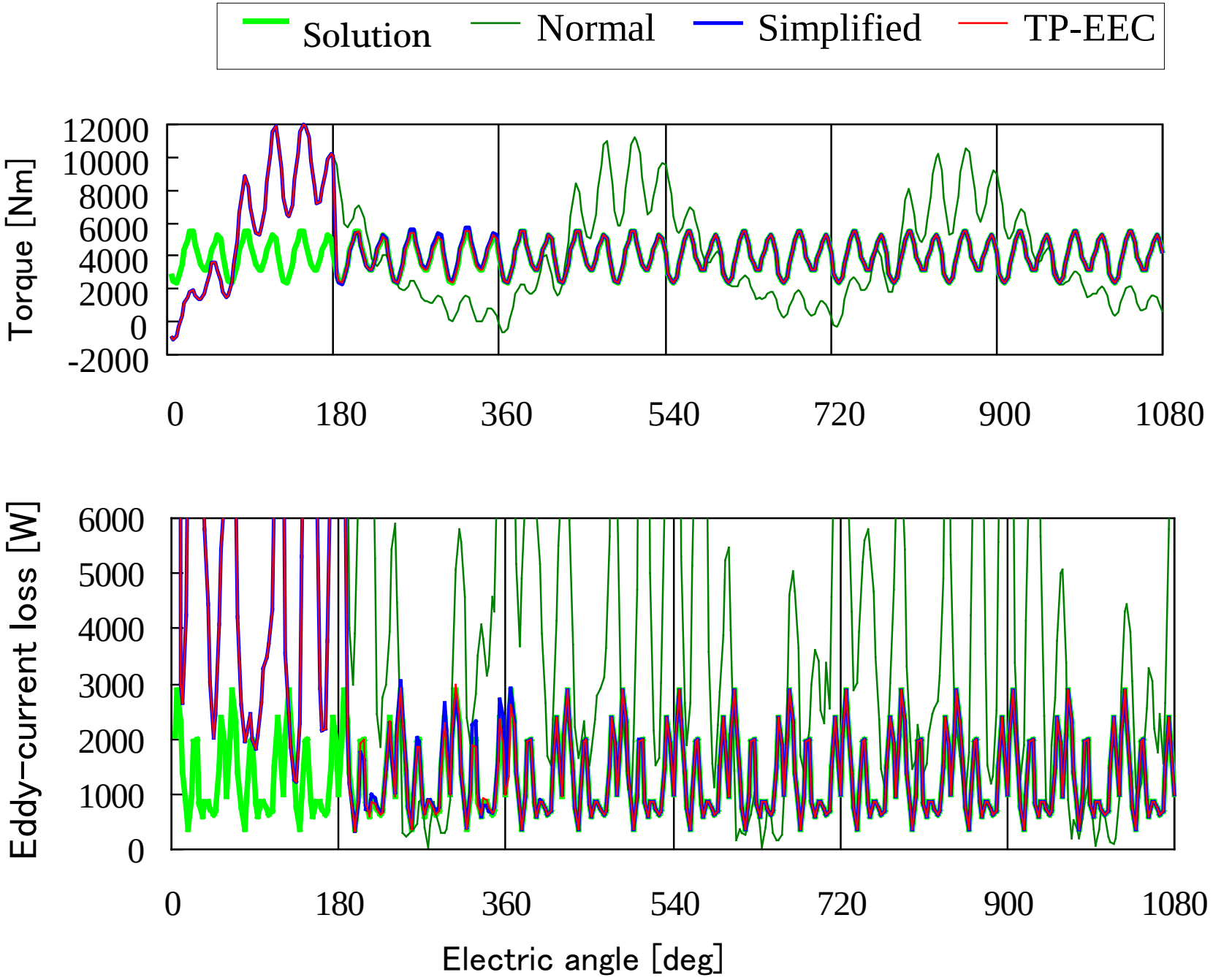
+

The rotor has ordinary time-periodicity.

simultaneously considered in the TP-EEC method

The simplified method is only applied to the stator

Methods	Elapsed time (s)
Normal step-by-step for 30 periods	Not converged
TP-EEC method	4167 (244)
Simplified TP-EEC method	4079 (242)


Solution
Normal
Simplified
TP-EEC
Torque [Nm]
12000
10000
8000
6000
4000
2000
0
-2000
0
180
360
540
720
900
1080
Eddy-current loss [W]
6000
5000
4000
3000
2000
1000
0
0
180
360
540
720
900
1080
Electric angle [deg]

Summary for part III

- The TP-EEC method was introduced to accelerate the convergence to the steady state solution in a transient analysis.
  - The TP-EEC method is based on the error correction method for the non-linear equation.
  - It focuses on the error correction of the DC component of the error.
  - Numerical tests confirms that the TP-EEC method improve the convergence.

Related work

- Parallel TP-EEC method was proposed.

(1) Each process/thread independently perform the step-by-step method. (Parallel in time)

(2) The continuity of the solution between processes/threads and the periodic boundary condition should be satisfied. The EEC framework for non-linear equation is applied for reducing the gap.

(3) The technique can be applied to any time evolution analysis (non time periodic problems).

(4) Parallel TP-EEC can be regarded as one of PARAREAL method.

Y. Takahashi, T. Tokumasu, K. Fujiwara, T. Iwashita, and H. Nakashima, "Parallel TP-EEC Method Based on Phase Conversion for Time-Periodic Nonlinear Magnetic Field Problems," IEEE Transactions on Magnetics, Vol. 51, No. 3, 2015.

Part IV Folded preconditioning

- 主たる研究者： 美舩健（京大）
- 共同研究者： 髙橋康人（同志社大）
- References
  - Takeshi Mifune, Yasuhito Takahashi and Takeshi Iwashita, “Folded Preconditioner: a New Class of Preconditioners for Krylov Subspace Methods to Solve Redundancy-Reduced Linear Systems of Equations”, IEEE Transactions on Magnetics, Vol. 45, No. 5, (2009), pp. 2068-2075.
  - Takeshi Mifune, Yasuhito Takahashi, and Takeshi Iwashita, “New Preconditioning Technique to Avoid Convergence Deterioration due to the Zero-Tree Gauge Condition in Magnetostatic Analysis”, IEEE Trans. Magn., Vol. 46, No. 7, (2010), pp. 2579-2584.
  - Yasuhito Takahashi, Takeshi Mifune, Takeshi Iwashita, Koji Fujiwara and Yoshiyuki Ishihara, “Folded IC Preconditioning in Quasi-Static Field Analysis Taking Account of Both Tree-Cotree and ϕ=0 Gauge Conditions”, IEEE Trans. Magn., Vol. 47, No. 5, (2011), pp. 1342-1345.

連立一次方程式

$$\boldsymbol{Ax} = \boldsymbol{b}$$

N: 方程式の未知数の数
A：係数行列　$(N \times N)$
$\boldsymbol{x}$：N次の解ベクトル（未知）
$\boldsymbol{b}$：N次の右辺ベクトル（既知）

※本資料では A が正方$(N \times N)$の場合のみに限定

反復法

$$A\boldsymbol{x} = \boldsymbol{b} \quad \left(A \in \boldsymbol{C}^{N \times N}, \quad \boldsymbol{x} \in \boldsymbol{C}^{N}, \quad \boldsymbol{b} \in \boldsymbol{C}^{N} \right)$$

- 第 i ステップの近似解： $\boldsymbol{x}^{(i)}$
  ※簡単のため、本資料では $\boldsymbol{x}^{(0)} = 0$ を仮定
- 決められた手順で $\boldsymbol{x}^{(i)}$ を更新する($i = 1, 2, \ldots$)
- 残差 $\boldsymbol{r}^{(i)} = \boldsymbol{b} - A\,\boldsymbol{x}^{(i)}$ が十分小さくなったら収束とみなす

クリロフ部分空間法

- クリロフ部分空間:

$$K^{(i)}\left(A; \boldsymbol{b}\right) = \mathrm{span}\left\{\boldsymbol{b}, A\boldsymbol{b}, \ldots, A^{i-1}\boldsymbol{b}\right\}$$

  - 第 i ステップの近似解を $K^{(i)}(A, \boldsymbol{b})$ から探索

$$\boldsymbol{x}^{(i)} \in K^{(i)}\left(A; \boldsymbol{b}\right)$$

つまり、

$$\boldsymbol{x}^{(1)} \in \mathrm{span}\left\{\boldsymbol{b}\right\}$$

$$\boldsymbol{x}^{(2)} \in \mathrm{span}\left\{\boldsymbol{b}, A\boldsymbol{b}\right\}$$

$$\boldsymbol{x}^{(3)} \in \mathrm{span}\left\{\boldsymbol{b}, A\boldsymbol{b}, A^2\boldsymbol{b}\right\}$$

$$\vdots$$

（※本資料では $\boldsymbol{x}^{(0)} = 0$ を仮定している）

前処理

- 前処理： 収束に必要な反復数を減らし、計算を高速化するための工夫

元の方程式 $A\boldsymbol{x} = \boldsymbol{b}$

$AM^{-1}M\boldsymbol{x} = \boldsymbol{b}$

（右）前処理された方程式 $A'\boldsymbol{x}' = \boldsymbol{b}$

M：前処理行列

$(N \times N)$

$A' = A\,M^{-1}$

$\boldsymbol{x}' = M\,\boldsymbol{x}$

前処理付きクリロフ部分空間法

- 各ステップの近似解を $K^{(i)}(A', \boldsymbol{b})$ から探索

$$\boldsymbol{x}'^{(i)} \in K^{(i)}\left(A'; \boldsymbol{b}\right) = \mathrm{span}\left\{\boldsymbol{b}, A'\boldsymbol{b}, \ldots, A'^{i-1}\boldsymbol{b}\right\}$$

$A' = AM^{-1}$, 元の方程式の解は $\boldsymbol{x}^{(i)} = M^{-1}\,\boldsymbol{x}'^{(i)}$ で求まるので,

$$\begin{aligned}\boldsymbol{x}^{(i)} &\in \mathrm{span}\left\{M^{-1}\boldsymbol{b}, M^{-1}AM^{-1}\boldsymbol{b}, \ldots, M^{-1}\left(AM^{-1}\right)^{i-1}\boldsymbol{b}\right\} \\ &\in K^{(i)}\left(M^{-1}A; M^{-1}\boldsymbol{b}\right)\end{aligned}$$

M が適切に選ばれていれば、反復収束性が改善される
（通常、$M \simeq A$ を目指す）

不定な連立一次方程式（A が非正則）

- $\det(A) = 0$
  $(\mathrm{rank}(A) < N)$

例：$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 解は $x_1 + x_2 = 1$ を満たすもの全て

- 解は無数に存在し一意に決まらない
  （解が存在しない場合はここでは扱わない）
- 電磁界解析では，しばしば現れる
  （無数の解のうち、ある一つの解を得ればよい）
  - curl curl の形（静磁界）の式の離散化
  - 辺・節点要素を併用する A–φ法

一般の不定な連立一次方程式

$$A\boldsymbol{x} = \boldsymbol{b}, \quad \mathrm{rank}(A) = L < N$$

未知数, 式の順序を適切に選べば, 一般に

$$\begin{array}{c} \\ L \\ N-L \end{array}\!\!\begin{array}{c} \overbrace{}^{L}\ \overbrace{}^{N-L} \\ \left(\begin{array}{c|c} A_{\mathrm{r}} & * \\ \hline * & * \end{array}\right)\left(\begin{array}{c} \boldsymbol{x}_{\mathrm{r}} \\ \hline \boldsymbol{x}_{\mathrm{ex}} \end{array}\right) = \left(\begin{array}{c} \boldsymbol{b}_{\mathrm{r}} \\ \hline * \end{array}\right) \end{array}\!\!\begin{array}{c} \\ L \\ N-L \end{array}$$

$A_{\mathrm{r}} \in C^{L\times L}$ は正則, $B \in C^{L\times(N-L)}$, $C \in C^{(N-L)\times L}$

$$C_{\mathrm{E}} A_{\mathrm{r}} B_{\mathrm{E}} \boldsymbol{x} = C_{\mathrm{E}} \boldsymbol{b}_{\mathrm{r}}$$

$$B_{\mathrm{E}} = \begin{pmatrix} E & B \end{pmatrix}, \quad C_{\mathrm{E}} = \begin{pmatrix} E \\ C \end{pmatrix} \qquad E \in C^{L\times L} \text{ は 単位行列}$$

未知数の数の削減

$$A_r x_r + A_r B x_{ex} = b_r$$

$$C A_r x_r + C A_r B x_{ex} = C b_r$$

下の式は上の式に C を乗じただけなので不要

$$A_r B_E x = b_r \quad \left(\because B_E x = x_r + B x_{ex} \right)$$

$y = B_E x \in C^L$ とおく

$$A_r y = b_r$$

L 次に縮小された方程式（y は一意）

y が求まれば, 元の式の解の一つが ,
$x_r \leftarrow y, x_{ex} \leftarrow 0$ で求まる

A-ϕ法による電磁波解析の場合

ベクトルポテンシャルとスカラポテンシャルを未知数として定式化（辺要素・節点要素を併用した有限要素法）

波動方程式

離散化

ベクトルポテンシャル
に対応する未知数

$$\begin{pmatrix} A_r & A_r B \\ CA_r & CA_r B \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_r \\ \boldsymbol{x}_{ex} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_r \\ C\boldsymbol{b}_r \end{pmatrix}$$

$$A_r \boldsymbol{y} = \boldsymbol{b}_r$$

未知数の数を削減
（A法）

スカラポテンシャルに
対応する未知数

未知数の消去は計算の高速化につながる？

- 電磁波解析の例題

	A-ϕ法	A法
未知数の数	483,887	363,107
行列の非零要素数（上三角部）	13,930,984	6,006,199
前処理－クリロフ部分空間法	IC－COCG	IC－COCG
メモリ消費 (MB)	663	356
収束までの反復回数	302	1772
求解時間 (s)	98.8	265.5

ここまでのまとめ

不定な方程式　$A\boldsymbol{x} = \boldsymbol{b}, \quad \mathrm{rank}(A) = L < N$

$$A = C_{\mathrm{E}} A_{\mathrm{r}} B_{\mathrm{E}}, \quad \boldsymbol{b} = C_{\mathrm{E}} \boldsymbol{b}_{\mathrm{r}} \quad \cdots\cdots\cdots (※1)$$

ただし，$B_{\mathrm{E}} = \begin{pmatrix} E & B \end{pmatrix} \in C^{L \times N}, C_{\mathrm{E}} = \begin{pmatrix} E \\ C \end{pmatrix} \in C^{N \times L}$

$A_{\mathrm{r}} \in C^{L \times L}$は正則，$E \in C^{L \times L}$は単位行列

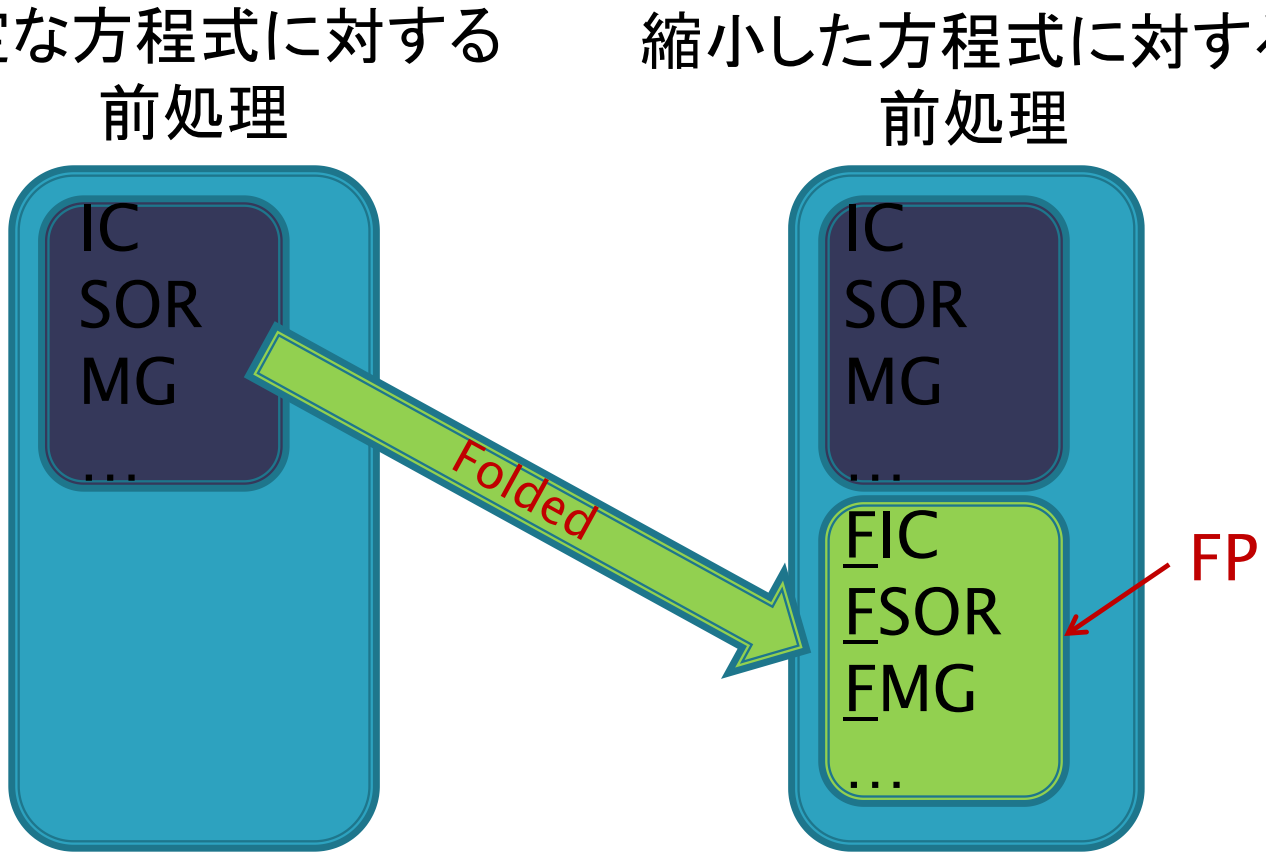
縮小した方程式　$A_{\mathrm{r}} \boldsymbol{y} = \boldsymbol{b}_{\mathrm{r}} \quad (\boldsymbol{y} = B_{\mathrm{E}} \boldsymbol{x})$

反復の収束が悪く，計算時間面でかえって不利に

（メモリは節約できる ）

新しい前処理 FP

- FP (Folded Preconditioning)



FP の定義

$A = C_{\mathrm{E}} A_{\mathrm{r}} B_{\mathrm{E}}$ についての前処理 Mに対して,
以下で定義される $L \times L$ 行列 M_{f} を使う
(A_{r}についての)前処理を FP と呼ぶ

$$M_{\mathrm{f}}^{-1} = B_{\mathrm{E}} M^{-1} C_{\mathrm{E}} \quad \cdots\cdots\cdots\cdots\cdots\cdots (※2)$$

FP の意味

不定な方程式(N 次)
$A\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{y} = B_{\mathrm{E}}\,\boldsymbol{x}$

縮小した方程式(L 次)
$A_{\mathrm{r}}\,\boldsymbol{y} = \boldsymbol{b}_{\mathrm{r}}$

ベースとする前処理付きクリロフ部分空間法
$\boldsymbol{x}^{(i)} \in K^{(i)}\left(M^{-1}A;\, M^{-1}\boldsymbol{b}\right)$

$\boldsymbol{y}^{(i)} = B_{\mathrm{E}}\,\boldsymbol{x}^{(i)}$

$\boldsymbol{y}^{(i)}$ は, 縮小した方程式の近似解とみなせる

1. $\boldsymbol{x}^{(i)}$ を求めるアルゴリズムと $\boldsymbol{y}^{(i)}$ を求めるアルゴリズムは, 元の方程式の残差の観点から, 同等（次スライド）
2. $\boldsymbol{y}^{(i)}$ を求める反復は, 縮小した方程式に対する前処理付きクリロフ部分空間法の形で得られる。ただしベースとした前処理に対応する FP を用いる（次々スライド）

元の方程式の残差に着目した等価性

$$\boldsymbol{r}^{(i)} = \boldsymbol{b} - A\boldsymbol{x}^{(i)}$$

$$= \boldsymbol{b} - C_E A_{\mathrm{r}} B_E \boldsymbol{x}^{(i)}$$

$$= \boldsymbol{b} - C_E A_{\mathrm{r}} B_E \begin{pmatrix} B_E \boldsymbol{x}^{(i)} \\ 0 \end{pmatrix} \qquad \left(\because B_{\mathrm{E}} = \begin{pmatrix} E & B \end{pmatrix}\right)$$

$$= \boldsymbol{b} - A \begin{pmatrix} \boldsymbol{y}^{(i)} \\ 0 \end{pmatrix} \qquad \cdots\cdots\cdots (※3)$$

$y^{(i)}$ のとりうる範囲

$$\boldsymbol{x}^{(i)} \in K^{(i)}\left(M^{-1}A; M^{-1}\boldsymbol{b}\right)$$

(※1)より

$$\boldsymbol{x}^{(i)} \in K^{(i)}\left(M^{-1}C_E A_{\mathrm{r}} B_E; M^{-1}C_E \boldsymbol{b}_{\mathrm{r}}\right)$$

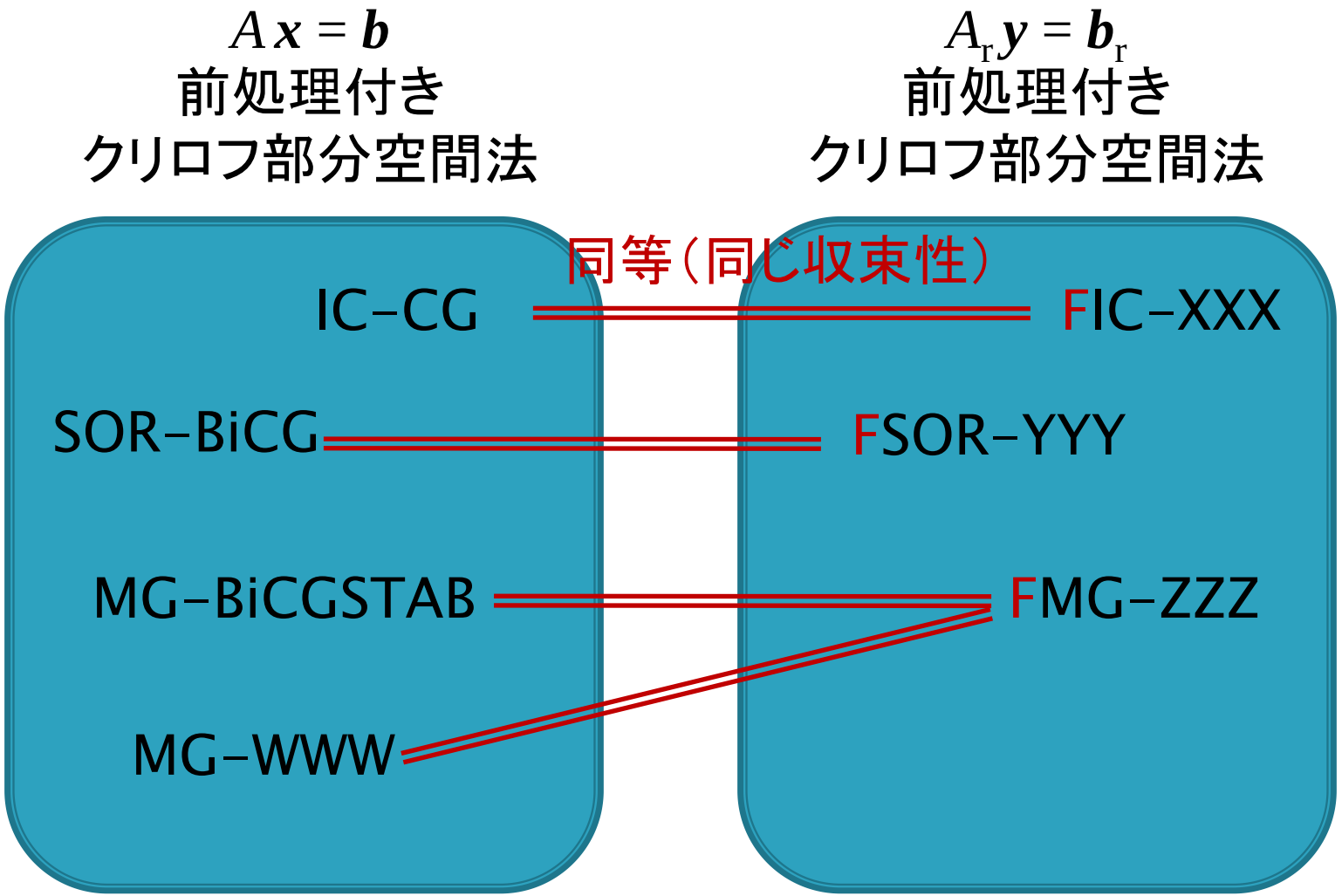
$$\boldsymbol{y}^{(i)} = B_E \boldsymbol{x}^{(i)} \in K^{(i)}\left(B_E M^{-1}C_E A_{\mathrm{r}}; B_E M^{-1}C_E \boldsymbol{b}_{\mathrm{r}}\right)$$

(※2)より

$$\boldsymbol{y}^{(i)} \in K^{(i)}\left(M_{\mathrm{f}}^{-1}A_{\mathrm{r}}; M_{\mathrm{f}}^{-1}\boldsymbol{b}_{\mathrm{r}}\right)$$

縮小した方程式に M_{f} を前処理として用いたときの探索空間に等しい

$A\boldsymbol{x} = \boldsymbol{b}$ に対する（任意の）前処理付きクリロフ部分空間法をベースとして，（※3）の意味で同等のアルゴリズムを，$A_{\mathrm{r}}\boldsymbol{y} = \boldsymbol{b}_{\mathrm{r}}$ に対する前処理付きクリロフ部分空間法の形で設計できる。ただし ベースとした前処理に対応する FP を用いる。


$Ax = b$
前処理付き
クリロフ部分空間法
$A_r y = b_r$
前処理付き
クリロフ部分空間法
同等（同じ収束性）
IC-CG
FIC-XXX
SOR-BiCG
FSOR-YYY
MG-BiCGSTAB
FMG-ZZZ
MG-WWW

FPを使用する際の探索アルゴリズムの選択

(1)ベースとした前処理付きクリロフ部分空間法と同等になる探索アルゴリズム（前スライドのXXX, YYY, ZZZ）を設計する

(2)既知の効率的に設計されたアルゴリズム（CG, BiCGSTABなど）を用いる （推奨）

- 特別な場合
  - ベースとする探索アルゴリズムがCG, BiCGなどの時、(1)と(2)が同じ結果になることもある

BiCG 法の場合

$x^{(i)}$ は，双直交条件を満たすよう決定される

$$\left(\boldsymbol{r}^{(i)}, \left(M^{-H}A^{H}\right)^{j}\boldsymbol{b}^{*}\right)=0, \quad 0 \le j < i$$

（$\boldsymbol{b}^{*}$ はシャドウベクトル）

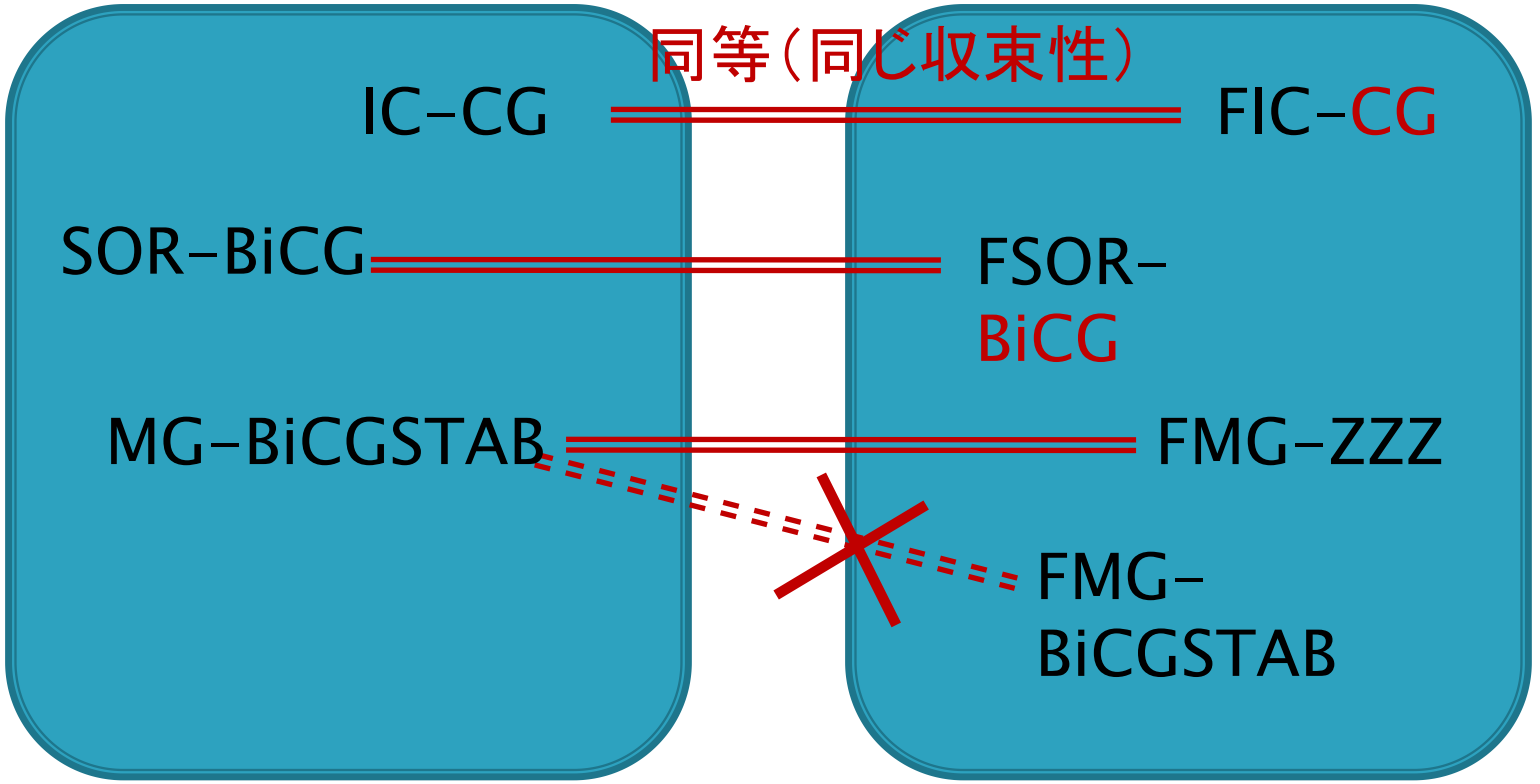
$$\left(C_{\mathrm{E}}\boldsymbol{r}_{\mathrm{r}}^{(i)}, \left(M^{-H}A^{H}\right)^{j}\boldsymbol{b}^{*}\right)=0, \quad 0 \le j < i$$

(※1) (※2)より

$$\left(\boldsymbol{r}_{\mathrm{r}}^{(i)}, \left(M_{\mathrm{f}}^{-H}A_{\mathrm{r}}^{H}\right)^{j}C_{\mathrm{E}}^{H}\boldsymbol{b}^{*}\right)=0, \quad 0 \le j < i$$

$Ax = b$
前処理付き
クリロフ部分空間法

$A_r y = b_r$
前処理付き
クリロフ部分空間法



（“同じ収束性”は保証されないが、性能は期待できる）

FP の実装と計算コスト

探索アルゴリズム(CGなど)の部分については，未知数を減らしたことで，計算コストの削減が期待できる

前処理の実装に必要なのは，行列ベクトル積： $\boldsymbol{z} = M_{\mathrm{f}}^{-1}\boldsymbol{w}$ のみ

$$M_{\mathrm{f}}^{-1} = B_{\mathrm{E}} M^{-1} C_{\mathrm{E}}$$

- M^{-1} についての積
  - ベースとした前処理のコストと同じ
- $B_{\mathrm{E}}, C_{\mathrm{E}}$ についての積
  - 効率よく計算できれば成功

A−ϕ 法の場合

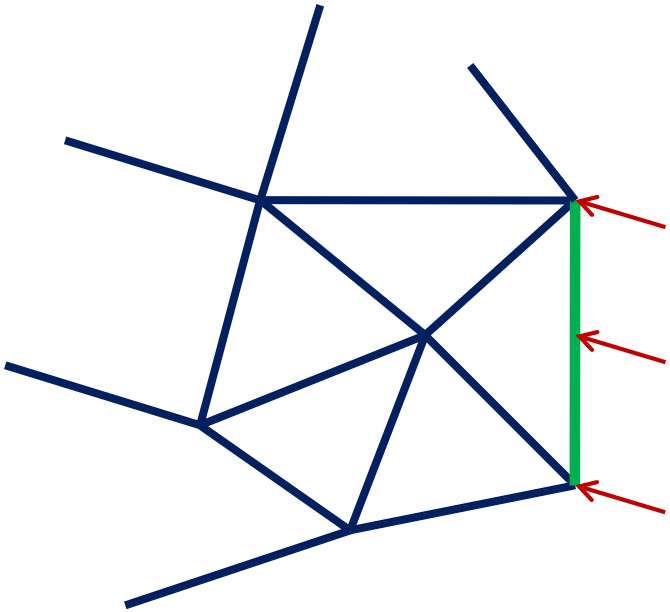
A−ϕ 法の方程式では,

$$B_{\mathrm{E}} = \begin{pmatrix} E & G \end{pmatrix}, \quad C_{\mathrm{E}} = \begin{pmatrix} E \\ G^T \end{pmatrix}$$

なので, G, G^T に関する積が速く計算できれば良い
ここで G は, 離散勾配行列（次スライド）

離散勾配行列 G

有限要素メッシュ



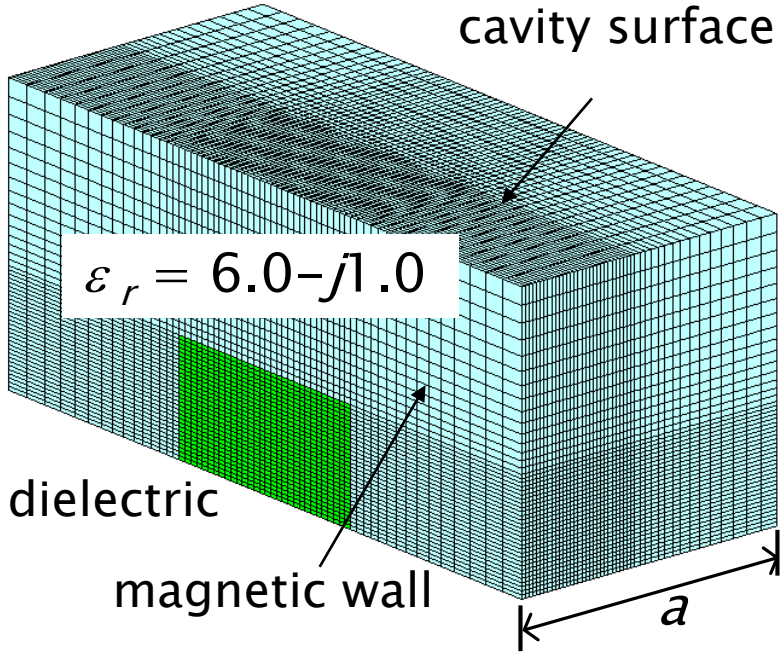
G：（辺数）×（節点数）行列

G の 第 e 行の非零成分は
$(e, n_1), (e, n_2)$ 成分
の2つのみ

G, G^T とベクトルの積は
十分高速に行える

数値計算例1

- 電磁波解析



要素数	121,500
節点数	129,076
ポート数	2
周波数	約6.2GHz ($k_0 a = 2.6$)

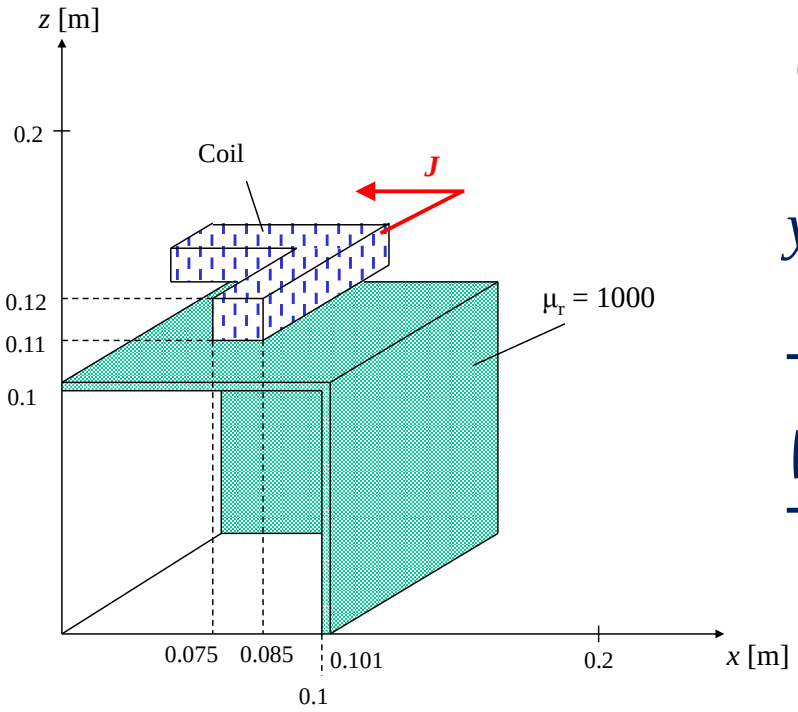
PC(Xeon X5472, 8GB RAM)

Formulations	A-ϕ	A
未知数の数	483,887	363,107
係数行列の非零要素数 (上三角部分)	13,930,984	6,006,199

	A-ϕ	A		
解法	IC－COCG	IC－COCG	FIC－COCG	FIC(改良版)－COCG
メモリ消費 (MB)	663	356	542	524
反復回数	302	1772	302	302
求解時間 (s)	98.8	265.5	89.4	81.8

数値計算例2

静磁界解析



直方体辺要素
70×70×70 分割
（辺数は 1,058,610）

yz, *zx* 平面は固定境界

Tree-Zero ゲージについては, *x* 軸に平行な辺全てを Tree として選択

PC(Core2 X9770, 8GB RAM)

Formulations	ゲージ無し	Tree-Zero ゲージ
未知数の数	1,058,610	690,900
係数行列の非零要素数 (上三角部分)	17,297,701	7,443,450

	ゲージ無し	Tree-Zero ゲージ	
解法	IC－CG	IC－CG	FIC－CG
メモリ消費 (MB)	496	240	**378**
反復回数	673	10000 以上	**679**
求解時間 (s)	134.1	―	**111.6**

Part IV のまとめ

- 非正則な係数行列を持つ連立一次方程式に対するFolded preconditioning(FP)を提案
- 同前処理をクリロフ部分空間法に適用した場合，収束性の劣化を伴わずに冗長な未知変数を削除できる
  - 一部の反復法では等価な収束性が得られる
- FPを使用する際のコストは反復前に評価できる